

# SOME REMARKS ON BRIDGELAND STABILITY CONDITIONS ON K3 AND ENRIQUES SURFACES

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ABSTRACT. We give some remarks on our papers with Minamide and Yanagida on Bridgeland stability conditions. We also give a remark on stability conditions on Enriques surfaces, and give another proof of the projectivity of the coarse moduli spaces of semi-stable objects, which were obtained by Nuer.

## 0. INTRODUCTION

In [3], Bridgeland introduced a very useful notion of stability condition on the derived category  $\mathbf{D}(X)$  of coherent sheaves on a projective scheme  $X$ , and showed that the set of stability conditions  $\text{Stab}(X)$  has a structure of complex manifold. A stability condition  $\sigma = (\mathcal{P}_\sigma, Z_\sigma)$  consists of an abelian category  $\mathcal{P}_\sigma$  which is a heart of a  $t$ -structure of  $\mathbf{D}(X)$  and a stability function  $Z_\sigma : \mathbf{D}(X) \rightarrow \mathbb{C}$  with some properties such as the Harder-Narasimhan property. If  $X$  is a K3 surface, a detailed description of a connected component of  $\text{Stab}(X)$  was given in [4]. In particular Bridgeland constructed a particular family of stability conditions so called *geometric stability conditions*: They are stability conditions such that  $\mathcal{O}_x$  ( $x \in X$ ) are stable with the same phase, and forms an open subset of  $\text{Stab}(X)$ . In [9], [10], we studied Fourier-Mukai transforms on K3 and abelian surfaces by using Bridgeland stability conditions. For this purpose, we constructed stability conditions on K3 surfaces by extending Bridgeland's construction of geometric stability conditions [4]. In this note, we give some remarks on our papers. We first add a remark on the relation between Bridgeland's construction and our extension. In [10], we constructed isomorphisms of the moduli stacks by using Fourier-Mukai transforms under some technical conditions. In this article, we shall remove one of the technical conditions. We also give a remark on stability conditions on Enriques surfaces, and prove that the coarse moduli spaces of semi-stable objects are projective schemes, which were obtained by Nuer [11].

## 1. PRELIMINARIES

**1.1. Notation.** Let  $X$  be a K3 surface over an algebraically closed field  $k$ . For  $E \in \mathbf{D}(X)$ , We denote the Mukai vector of  $E$  by

$$v(E) = \text{ch}(E)\sqrt{\text{td}_X} = \text{rk } E + c_1(E) + (\text{ch}_2(E) + \text{rk } E \varrho_X),$$

where  $\varrho_X$  is the fundamental class of  $X$ . We also set

$$(r, \xi, a) = r + \xi + a\varrho_X, \xi \in \text{NS}(X), a \in \mathbb{Q}.$$

Let  $(H^*(X, \mathbb{Z}), \langle \cdot, \cdot \rangle)$  be the Mukai lattice of  $X$ , where

$$\langle v_1, v_2 \rangle := (\xi_1 \cdot \xi_2) - r_1 a_2 - r_2 a_1, v_i = (r_i, \xi_i, a_i), i = 1, 2.$$

We shall give some notation on stability conditions and also some properties. For more details, see [3] and [4]. For a stability condition  $\sigma = (\mathcal{P}_\sigma, Z_\sigma)$ ,  $\phi_\sigma$  is the phase function and  $\mathcal{P}_\sigma(\phi)$  denotes the set of  $\sigma$ -semi-stable objects  $E$  with  $\phi_\sigma(E) = \phi$ .  $\mathcal{P}_\sigma(I)$  denotes the category generated by objects  $E \in \cup_{\phi \in I} \mathcal{P}_\sigma(\phi)$ . For  $E \in \mathbf{D}(X)$ ,  $\phi_\sigma^+(E)$  is the maximum of the stable factors of  $E$  and  $\phi_\sigma^-(E)$  the minimum of the stable factors of  $E$ .

Let  $\mathcal{P}(X)$  be the subset of  $v(K(X))_{\mathbb{C}}$  consisting of  $\mathfrak{U}$  such that  $\text{Re } \mathfrak{U}$  and  $\text{Im } \mathfrak{U}$  span a positive definite 2-plane in  $v(K(X))_{\mathbb{R}}$ . We shall regard  $\mathcal{P}(X)$  as a subset of  $\text{Hom}(v(K(X)), \mathbb{C})$  by  $\langle \mathfrak{U}, \bullet \rangle$ . Let  $\mathcal{P}^+(X)$  be the connected component of  $\mathcal{P}(X)$  containing  $e^{\beta+i\omega}$ , where  $\omega$  is ample. Let  $P^+(X)_{\mathbb{R}}$  be the positive cone of  $X$  and  $\text{Amp}(X)_{\mathbb{R}} (\subset P^+(X)_{\mathbb{R}})$  the ample cone of  $X$ . We have an action of  $GL_2^+(\mathbb{R})$  on  $\text{Hom}(v(K(X)), \mathbb{C})$  and  $\mathcal{P}^+(X)/GL_2^+(\mathbb{R}) = \text{NS}(X)_{\mathbb{R}} \times P^+(X)_{\mathbb{R}}$ . Let  $\Delta(X)$  be the set of Mukai vectors  $u$  with  $\langle u^2 \rangle = -2$ . We set

$$\mathcal{P}_0^+(X) := \mathcal{P}^+(X) \setminus \cup_{u \in \Delta(X)} u^\perp.$$

For  $(\beta, \omega) \in \text{NS}(X)_{\mathbb{R}} \times \text{Amp}(X)_{\mathbb{R}}$  with  $\langle e^{\beta+i\omega}, u \rangle \notin \mathbb{R}_{\leq 0}$  ( $u \in \Delta(X)$ ), Bridgeland constructed a stability condition  $\sigma_{(\beta, \omega)} = (\mathcal{A}_{(\beta, \omega)}, Z_{(\beta, \omega)})$  such that  $Z_{(\beta, \omega)}(\bullet) = \langle e^{\beta+i\omega}, \bullet \rangle$  and  $\mathcal{O}_x$  ( $x \in X$ ) are stable objects. Up to the action of  $\widetilde{GL}_2^+(\mathbb{R})$  on  $\text{Stab}(X)$ , it is characterized as a stability condition  $\sigma$  such that  $\mathcal{O}_x$  ( $x \in X$ ) are  $\sigma$ -stable objects with a fixed phase and  $Z_\sigma \in \mathcal{P}^+(X)$ . Let  $U(X)$  be the open subset of  $\text{Stab}(X)$

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consisting of these stability conditions. Let  $\text{Stab}^\dagger(X)$  be the connected component containing  $U(X)$ . Then  $\text{Stab}^\dagger(X) \rightarrow \mathcal{P}_0^+(X)$  is a covering map.

For a Mukai vector  $v$ ,  $\mathcal{M}_\sigma(v)$  denotes the moduli stack of  $\sigma$ -semi-stable objects  $E$  with  $v(E) = v$  and  $M_\sigma(v)$  denotes the coarse moduli scheme of  $S$ -equivalence classes of  $\sigma$ -semi-stable objects  $E$  with  $v(E) = v$ . If  $\sigma = \sigma_{(\beta, \omega)}$ , then we set  $\mathcal{M}_{(\beta, \omega)}(v) := \mathcal{M}_{\sigma_{(\beta, \omega)}}(v)$  and  $M_{(\beta, \omega)}(v) := M_{\sigma_{(\beta, \omega)}}(v)$ .

*Remark 1.1.* For a  $\sigma$ -semi-stable object  $E$  with  $v(E) = v$ ,  $\phi_\sigma(E) \bmod 2$  is determined by  $Z_\sigma(v)$ . Although we need to fix  $\phi_\sigma(E)$  for precise definitions, we adopt the above definitions.

**1.2. Stability conditions associated to a category of perverse coherent sheaves.** Let us briefly recall our construction of  $\sigma_{(\beta, \omega)}$  for a nef and big divisor  $\omega$  in [9]. Let  $\pi : X \rightarrow Y$  be the minimal resolution of a normal K3 surface  $Y$ . Let  $H$  be the pull-back of an ample divisor on  $Y$  and  $(\beta \cdot C) \notin \mathbb{Z}$  for all exceptional  $(-2)$ -curves. Then there is a category of perverse coherent sheaves  $\mathfrak{C}$  with a local projective generator  $G$  on  $X$  such that  $c_1(G)/\text{rk } G = \beta$  [16, Prop. 2.4.5]: Thus there is a locally free sheaf  $G$  with  $c_1(G)/\text{rk } G = \beta$  such that

$$(1.1) \quad \begin{aligned} S &:= \{E \in \text{Coh}(X) \mid \pi_*(G^\vee \otimes E) = 0\}, \\ T &:= \{E \in \text{Coh}(X) \mid R^1\pi_*(G^\vee \otimes E) = 0\} \end{aligned}$$

is a torsion pair  $(T, S)$  of  $\text{Coh}(X)$  and the tilting is our category of perverse coherent sheaves:

$$(1.2) \quad \mathfrak{C} = \{E \in \mathbf{D}(X) \mid H^i(E) = 0, i \neq -1, 0, H^{-1}(E) \in S, H^0(E) \in T\}.$$

Here we would like to remark that  $R^1\pi_*(\mathcal{O}_X) = 0$  and [16, Assumption 1.1.1] holds.

*Remark 1.2.* Let  $\mathcal{D}$  be a connected component of  $\text{NS}(X)_\mathbb{R} \setminus \bigcup_{n, C} \{x \mid (x \cdot C) = n\}$ . Then  $(S, T)$  depends only on  $\mathcal{D}$  containing  $\beta$ . In particular,  $\mathfrak{C}$  is well-defined even when  $\beta$  is not defined over  $\mathbb{Q}$ .

*Remark 1.3.* If  $\pi$  is an isomorphism, then  $H$  is ample and  $\mathfrak{C} = \text{Coh}(X)$ .

**Definition 1.4.** (1) For  $E \in \mathbf{D}(X)$ ,  ${}^pH^i(E) \in \mathfrak{C}$  denotes the  $i$ -th cohomology object of  $E$  with respect to the  $t$ -structure defining  $\mathfrak{C}$ .

(2) For a morphism  $\psi : E \rightarrow F$  in  $\mathfrak{C}$ ,  $\ker_{\mathfrak{C}} \psi$ ,  $\text{im}_{\mathfrak{C}} \psi$  and  $\text{coker}_{\mathfrak{C}} \psi$  denote the kernel, the image and the cokernel of  $\psi$  in  $\mathfrak{C}$  respectively.

**Definition 1.5.** For  $E \in \mathfrak{C}$ , we define the *dimension*  $\dim E$  of  $E$  by

$$\dim E := \dim \pi(\text{Supp}(H^{-1}(E)) \cup \text{Supp}(H^0(E))).$$

**Definition 1.6.** For  $\beta \in \text{NS}(X)_\mathbb{R}$  and  $\omega \in \overline{\text{Amp}(X)}_\mathbb{R}$  with  $(\omega^2) > 0$ , we set  $\deg_\beta(E) := (c_1(E(-\beta)) \cdot \omega)$  and  $\chi_\beta(E) := \chi(E(-\beta))$ . We also set  $a_\beta(E) := -\langle e^\beta, v(E) \rangle$ .

For a local projective generator  $G$  of  $\mathfrak{C}$  and a perverse coherent sheaf  $E \in \mathfrak{C}$ , we have the  $G$ -twisted Hilbert polynomial  $\chi(G, E(nH))$  with respect to  $H$ . The degree of the  $G$ -twisted Hilbert polynomial of  $E$  is  $\dim E$ . By using the  $G$ -twisted Hilbert polynomial, we have a notion of semi-stability as in the Gieseker semi-stability for ordinary coherent sheaves.

**Definition 1.7** ([16, Defn. 1.4.1]). Let  $\mathfrak{C}$  be a category of perverse coherent sheaves on  $X$ ,  $G$  a local projective generator of  $\mathfrak{C}$ , and  $E \in \mathfrak{C}$  a perverse coherent sheaf.

- (i) (a) If  $\dim E \leq 1$ , then  $E$  is called a *torsion object*.
- (b) If there is no subobject  $F \neq 0$  with  $\dim F < d = \dim E$ , then  $E$  is called *purely  $d$ -dimensional*. In particular, if  $E$  is purely 2-dimensional, then  $E$  is called *torsion free*.
- (ii) A 2-dimensional object  $E$  is  *$G$ -twisted semi-stable with respect to  $H$*  if

$$(1.3) \quad \chi(G, F(nH)) \leq \frac{\text{rk } F}{\text{rk } E} \chi(G, E(nH)), \quad n \gg 0$$

for all proper subobjects  $F \neq 0$  of  $E$ . We also say a torsion free object  $E$  is  *$\mu$ -semi-stable* if

$$(1.4) \quad \frac{(c_1(F), H)}{\text{rk } F} \leq \frac{(c_1(E), H)}{\text{rk } E}$$

for all subobjects  $F$  of  $E$  with  $0 < \text{rk } F < \text{rk } E$ .

- (iii) If  $E$  is 1-dimensional, then  $E$  is  *$G$ -twisted semi-stable with respect to  $H$*  if

$$(1.5) \quad \chi(G, F) \leq \frac{(H, c_1(F))}{(H, c_1(E))} \chi(G, E)$$

for all proper subobjects  $F \neq 0$  of  $E$ .

- (iv) Let  $\mathcal{M}_H^\gamma(v)$  denote the moduli stack of  $\gamma$ -twisted semi-stable objects  $E$  with  $v(E) = v$ , and  $M_H^\gamma(v)$  the coarse moduli scheme of  $S$ -equivalence classes of  $\gamma$ -twisted semi-stable objects [16, sect. 1.4].

**Definition 1.8.** (1) For  $(\beta, \omega) \in \text{NS}(X)_{\mathbb{R}} \times P^+(X)_{\mathbb{R}}$  and  $E \in \mathbf{D}(X)$ , we set

$$Z_{(\beta, \omega)}(E) := \langle e^{\beta+i\omega}, v(E) \rangle.$$

(2) Under the assumption  $e^{\beta+i\omega} \notin \cup_{u \in \Delta(X)} u^{\perp}$  with  $(\beta, \omega) \in \text{NS}(X)_{\mathbb{Q}} \times \pi^*(\text{Amp}(Y)_{\mathbb{Q}})$ , we define a torsion pair  $(\mathcal{T}_{(\beta, \omega)}, \mathcal{F}_{(\beta, \omega)})$  of  $\mathfrak{C}$  as follows:

- (a)  $\mathcal{T}_{(\beta, \omega)}$  is the full subcategory of  $\mathfrak{C}$  consisting of  $E$  such that  $Z_{(\beta, \omega)}(F) \in \mathbb{H} \cup \mathbb{R}_{<0}$  for any quotient  $E \rightarrow F (\neq 0)$  of  $E$ .
- (b)  $\mathcal{F}_{(\beta, \omega)}$  is the full subcategory of  $\mathfrak{C}$  consisting of  $E$  such that  $-Z_{(\beta, \omega)}(F) \in \mathbb{H} \cup \mathbb{R}_{<0}$  for any subobject  $(0 \neq) F \rightarrow E$  of  $E$ .

Let  $\mathcal{A}_{(\beta, \omega)}$  be the tilting of the torsion pair  $(\mathcal{T}_{(\beta, \omega)}, \mathcal{F}_{(\beta, \omega)})$ .

The definition of  $(\mathcal{T}_{(\beta, \omega)}, \mathcal{F}_{(\beta, \omega)})$  is equivalent to the definition in [9, Defn. 1.5.7].

For  $(\beta, \omega) \in \text{NS}(X)_{\mathbb{R}} \times \pi^*(\text{Amp}(Y)_{\mathbb{R}})$ , we also define a pair of subcategories  $(\mathcal{T}_{(\beta, \omega)}^*, \mathcal{F}_{(\beta, \omega)}^*)$  of  $\text{Coh}(X)$  as follows:

- (i)  $\mathcal{T}_{(\beta, \omega)}^*$  is the full subcategory of  $\text{Coh}(X)$  consisting of  $E$  such that  $Z_{(\beta, \omega)}(F) \in \mathbb{H} \cup \mathbb{R}_{<0}$  for any quotient  $E \rightarrow F (\neq 0)$  of  $E$ .
- (ii)  $\mathcal{F}_{(\beta, \omega)}^*$  is the full subcategory of  $\text{Coh}(X)$  consisting of  $E$  such that  $-Z_{(\beta, \omega)}(F) \in \mathbb{H} \cup \mathbb{R}_{<0}$  for any subsheaf  $(0 \neq) F \rightarrow E$  of  $E$ .

The relation of these definitions are given by the following proposition.

**Proposition 1.9.** Assume that  $\beta$  and  $\omega$  are defined over  $\mathbb{Q}$ .

- (1) For  $E \in \mathcal{A}_{(\beta, \omega)}$ ,  $H^i(E) = 0$ , ( $i \neq -1, 0$ ).
- (2) For  $E \in \text{Coh}(X)$ , there is an exact sequence

$$(1.6) \quad 0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$$

such that  $E_1 \in \mathcal{T} := \mathcal{A}_{(\beta, \omega)} \cap \text{Coh}(X)$  and  $E_2 \in \mathcal{F} := \mathcal{A}_{(\beta, \omega)}[-1] \cap \text{Coh}(X)$ . Thus  $(\mathcal{T}, \mathcal{F})$  is a torsion pair of  $\text{Coh}(X)$ .

- (3)  $(\mathcal{T}, \mathcal{F}) = (\mathcal{T}_{(\beta, \omega)}^*, \mathcal{F}_{(\beta, \omega)}^*)$ . In particular,  $(\mathcal{T}_{(\beta, \omega)}^*, \mathcal{F}_{(\beta, \omega)}^*)$  is a torsion pair whose tilting is  $\mathcal{A}_{(\beta, \omega)}$ .

*Proof.* Let  $(T, S)$  be the torsion pair in (1.1). (1) For  $E \in \mathcal{A}_{(\beta, \omega)}$ , we see that  ${}^p H^{-1}(E)$  is a torsion free object of  $\mathfrak{C}$ , which implies  $H^{-1}({}^p H^{-1}(E)) = 0$ . Then  $H^i(E) = 0$  for  $i \neq -1, 0$ ,  $H^0({}^p H^0(E)) = H^0(E)$  and we have an exact sequence in  $\text{Coh}(X)$

$$0 \rightarrow {}^p H^{-1}(E) \rightarrow H^{-1}(E) \rightarrow H^{-1}({}^p H^0(E)) \rightarrow 0.$$

(2) For  $E \in \text{Coh}(X)$ , we have a decomposition

$$0 \rightarrow E_T \rightarrow E \rightarrow E_S \rightarrow 0$$

such that  $E_T \in T$  and  $E_S \in S$ . Since  $E_S[1] \in \mathfrak{C}$  is a 0-dimensional object,  $E_S[1] \in \mathcal{T}_{(\beta, \omega)} \subset \mathcal{A}_{(\beta, \omega)}$ . We also have a decomposition

$$0 \rightarrow E_1 \rightarrow E_T \rightarrow E'_1 \rightarrow 0$$

such that  $E_1 \in \mathcal{T}_{(\beta, \omega)}$  and  $E'_1 \in \mathcal{F}_{(\beta, \omega)}$ . Then  $E_2 := \text{Cone}(E_1 \rightarrow E) \in \mathcal{A}_{(\beta, \omega)}[-1]$ . By (1), we see that  $E_1, E_2 \in \text{Coh}(X)$ . Hence we have a decomposition (1.6). Since  $\text{Hom}(A, B) = 0$  for  $A \in \mathcal{T}$  and  $B \in \mathcal{F}$ ,  $(\mathcal{T}, \mathcal{F})$  is a torsion pair.

(3) We first prove that  $\mathcal{T} = \mathcal{T}_{(\beta, \omega)}^*$ . We note that  $\mathcal{T}_{(\beta, \omega)} \cap \text{Coh}(X) = \mathcal{T}$ . Let  $E$  be an element of  $\mathcal{T}_{(\beta, \omega)}^*$ . For  $F \in S$  and a non-zero homomorphism  $\psi : E \rightarrow F$ ,  $\text{im } \psi \in S$  and  $Z_{(\beta, \omega)}(\text{im } \psi) = -\chi_{\beta}(\text{im } \psi) > 0$ . Hence  $\text{Hom}(E, F) = 0$  for  $F \in S$  and  $E \in T$ . Let  $\psi : E \rightarrow F (\neq 0)$  be a quotient in  $\mathfrak{C}$ . Then we have an exact sequence in  $\text{Coh}(X)$

$$0 \rightarrow H^{-1}(F) \rightarrow H^0(\ker_{\mathfrak{C}} \psi) \rightarrow H^0(E) \rightarrow H^0(F) \rightarrow 0.$$

Since  $Z_{(\beta, \omega)}(H^0(F)) \in \mathbb{H} \cup \mathbb{R}_{<0}$  or  $H^0(F) = 0$ , and  $-Z_{(\beta, \omega)}(H^{-1}(F)) \in \mathbb{R}_{<0}$  or  $H^{-1}(F) = 0$ , we have  $Z_{(\beta, \omega)}(F) \in \mathbb{H} \cup \mathbb{R}_{<0}$ . Hence  $E \in \mathcal{T}_{(\beta, \omega)} \cap \text{Coh}(X) = \mathcal{T}$ . Conversely if  $E \in \mathcal{T} = \text{Coh}(X) \cap \mathcal{T}_{(\beta, \omega)}$ , then for any quotient  $\psi : E \rightarrow F (\neq 0)$  in  $\text{Coh}(X)$ ,  $\ker \psi$  in  $\text{Coh}(X)$  has a decomposition

$$0 \rightarrow E_1 \rightarrow \ker \psi \rightarrow E_2 \rightarrow 0$$

such that  $E_1 \in T$  and  $E_2 \in S$ . Then  $E_1 = \ker_{\mathfrak{C}} \psi$ ,  $E/E_1 = \text{im}_{\mathfrak{C}} \psi$  and  $\text{coker}_{\mathfrak{C}} \psi = E_2[1]$ . Hence  $Z_{(\beta, \omega)}(F) = Z_{(\beta, \omega)}(\text{im}_{\mathfrak{C}} \psi) - Z_{(\beta, \omega)}(E_2) \in \mathbb{H} \cup \mathbb{R}_{<0}$ . Therefore  $E \in \mathcal{T}_{(\beta, \omega)}^*$ , and we get  $\mathcal{T} = \mathcal{T}_{(\beta, \omega)}^*$ .

We next prove that  $\mathcal{F} = \mathcal{F}_{(\beta, \omega)}^*$ . Let  $E$  be an element of  $\mathcal{F}_{(\beta, \omega)}^*$ . We take a decomposition

$$0 \rightarrow E_T \rightarrow E \rightarrow E_S \rightarrow 0$$

such that  $E_T \in T$  and  $E_S \in S$ . Then  $E_S[1] \in \mathcal{T}_{(\beta, \omega)}$ . For a non-zero subobject  $\psi : F \rightarrow E_T$  of  $E_T$  in  $\mathfrak{C}$ ,  $F \in \text{Coh}(X)$  and we have an exact sequence

$$0 \rightarrow H^{-1}(\text{coker}_{\mathfrak{C}} \psi) \rightarrow H^0(F) \xrightarrow{\varphi} H^0(E_T) \rightarrow H^0(\text{coker}_{\mathfrak{C}} \psi) \rightarrow 0,$$

where  $H^{-1}(\text{coker}_{\mathfrak{C}} \psi) \in S$ . Since  $E_T \in \mathcal{F}_{(\beta, \omega)}^*$ ,  $-Z_{(\beta, \omega)}(\text{im } \varphi) \in \mathbb{H} \cup \mathbb{R}_{<0}$ . Hence  $-Z_{(\beta, \omega)}(H^0(F)) \in \mathbb{H} \cup \mathbb{R}_{<0}$ , which implies  $E_T \in \mathcal{F}_{(\beta, \omega)}$ . Hence  $E \in \mathcal{A}_{(\beta, \omega)}[-1] \cap \text{Coh}(X) = \mathcal{F}$ . For  $E \in \mathcal{A}_{(\beta, \omega)}[-1] \cap \text{Coh}(X) = \mathcal{F}$ , we have a decomposition

$$0 \rightarrow E_T \rightarrow E \rightarrow E_S \rightarrow 0$$

such that  $E_T \in T$  and  $E_S \in S$ . Obviously  $S \subset \mathcal{F}_{(\beta, \omega)}^*$ . Since  ${}^p H^0(E) \in \mathcal{F}_{(\beta, \omega)}$  and  ${}^p H^0(E) = E_T$ ,  $E_T \in \mathcal{F}_{(\beta, \omega)}$ . For a subsheaf  $F$  of  $E$ , we take a decomposition

$$0 \rightarrow F_T \rightarrow F \rightarrow F_S \rightarrow 0$$

such that  $F_T \in T$  and  $F_S \in S$ . Then  $F_T$  is a subsheaf of  $E_T$ . Moreover  $F_T \rightarrow E_T$  is injective in  $\mathfrak{C}$  by  $E_T/F_T \in T$ . Since  $E_T \in \mathcal{F}_{(\beta, \omega)}$ ,  $-Z_{(\beta, \omega)}(F_T) \in \mathbb{H} \cup \mathbb{R}_{<0}$  unless  $F_T = 0$ . We note that  $-Z_{(\beta, \omega)}(F_S) \in \mathbb{R}_{<0}$  unless  $F_S = 0$ . Hence  $-Z_{(\beta, \omega)}(F) \in \mathbb{H} \cup \mathbb{R}_{<0}$ . Thus  $E \in \mathcal{F}_{(\beta, \omega)}^*$ , which implies  $\mathcal{F} = \mathcal{F}_{(\beta, \omega)}^*$ .  $\square$

**Proposition 1.10** ([9]).  $\sigma_{(\beta, \omega)} := (\mathcal{A}_{(\beta, \omega)}, Z_{(\beta, \omega)})$  is a stability condition.

**Definition 1.11.** If  $(\mathcal{T}_{(\beta, \omega)}^*, \mathcal{F}_{(\beta, \omega)}^*)$  is a torsion pair of  $\text{Coh}(X)$ , then we also denote the tilting by  $\mathcal{A}_{(\beta, \omega)}$  and set  $\sigma_{(\beta, \omega)} := (\mathcal{A}_{(\beta, \omega)}, Z_{(\beta, \omega)})$ .

By Proposition 1.9,  $\sigma_{(\beta, \omega)}$  is the same stability condition in [9] if  $\beta, \omega$  are defined over  $\mathbb{Q}$ .

## 2. STABILITY CONDITIONS ON THE BOUNDARY OF THE GEOMETRIC CHAMBER

**2.1. Relation of stability conditions.** Let  $\pi : X \rightarrow Y$  be the minimal resolution of a normal K3 surface  $Y$ . For  $(\beta_0, \omega_0) := (\beta_0, tH)$ ,  $\sigma_{(\beta_0, \omega_0)} := (\mathcal{A}_{(\beta_0, \omega_0)}, Z_{(\beta_0, \omega_0)})$  denotes the stability condition constructed in subsection 1.2. For these stability conditions,  $\mathcal{O}_x$  are semi-stable objects with phase 1. We shall remark the relation of these stability conditions to stability conditions such that  $\mathcal{O}_x$  are stable.

We note that  $\sigma_{(\beta_0, \omega_0)}$  satisfies the support property [1], since the Bogomolov inequality holds. In particular, we have a wall and chamber structure. There is a small neighborhood  $B$  of  $(\beta_0, \omega_0)$  in  $\text{NS}(X)_{\mathbb{R}} \times P^+(X)_{\mathbb{R}}$  and a continuous map

$$(2.1) \quad \mathfrak{s} : B \rightarrow \text{Stab}(X)$$

such that  $Z_{\mathfrak{s}(\beta, \omega)} = Z_{(\beta, \omega)}$  and  $\mathfrak{s}(\beta_0, \omega_0) = \sigma_{(\beta_0, \omega_0)}$  (see also [4, Prop. 8.3]).

Let  $S$  be the set of stable factor  $E (\neq \mathcal{O}_x)$  of  $\mathcal{O}_x$  with respect to  $\sigma \in \mathfrak{s}(B)$ . Since  $\{\mathcal{O}_x \mid x \in X\}$  has bounded mass, [4, Lem. 9.2] implies  $W := \{v(E) \mid E \in S\}$  is a finite set. Let  $S'$  be a subset of  $S$  such that  $E_1 \in S'$  if and only if

- (i)  $E_1$  is a stable factor of  $\mathcal{O}_x$  with respect to  $\sigma \in \mathfrak{s}(B)$  and
- (ii)  $\phi_{\sigma}(E_1) = \phi_{\sigma}^+(\mathcal{O}_x)$ .

We set  $W' := \{v(E) \in W \mid E \in S', \text{rk } E > 0\}$ .

**Lemma 2.1.** Assume that  $Z_{\sigma_{(\beta_0, \omega_0)}}(E) \in \mathbb{R}_{<0}$  for all  $E \in S'$ . We set

$$B' := \{(\beta, \omega) \in B \mid \omega \in \text{Amp}(X), \text{Im } Z_{(\beta, \omega)}(w) > 0, w \in W'\}.$$

(1) Then  $B' \neq \emptyset$  and  $(\beta_0, \omega_0) \in \overline{B'}$ .

(2) Let

$$(2.2) \quad 0 \rightarrow E_1 \rightarrow \mathcal{O}_x \rightarrow E_2 \rightarrow 0$$

be an exact sequence in  $\mathcal{A}_{(\beta_0, \omega_0)}$  such that  $E_1 \in S'$ . Then  $\text{Im } Z_{(\beta, \omega)}(E_1) > 0$  for  $(\beta, \omega) \in B'$ .

*Proof.* Let  $w = r + \xi + a\varrho_X$  be an element of  $W'$ . For  $x, y \in \mathbb{R}_{>0}$  and  $\eta \in \omega_0^\perp$  with  $(\xi - r\beta_0, \eta) + rxy(\omega_0^2) > 0$ ,  $\text{Im } Z_{(\beta_0 - x\omega_0, y\omega_0 + \eta)}(w) > 0$ . We can take a small  $\eta$  such that  $y\omega_0 + \eta$  is ample. Therefore (1) holds.

(2) We note that  ${}^p H^{-1}(E_1) = 0$  and there is an exact sequence in  $\mathfrak{C}$

$$(2.3) \quad 0 \rightarrow {}^p H^{-1}(E_2) \rightarrow {}^p H^0(E_1) \xrightarrow{\psi} \mathcal{O}_x \rightarrow {}^p H^0(E_2) \rightarrow 0.$$

Hence  $\text{rk } E_1 \geq 0$ . If  $\text{rk } E_1 > 0$ , then  $v(E_1) \in W'$  implies  $\text{Im } Z_{\sigma}(E_1) > 0$  for  $\sigma \in \mathfrak{s}(B')$ . Assume that  $\text{rk } E_1 = 0$ . Since  $E_1, E_2$  are semi-stable objects with  $\phi_{\sigma_{(\beta_0, \omega_0)}}(E_1) = \phi_{\sigma_{(\beta_0, \omega_0)}}(E_2)$ ,  $E_1$  is a 0-dimensional object of  $\mathfrak{C}$ . Since  ${}^p H^{-1}(E_2)$  is torsion free in  $\mathfrak{C}$ ,  ${}^p H^{-1}(E_2) = 0$ . Since  $H^{-1}(\mathcal{O}_x) = 0$ , we get  $H^{-1}({}^p H^0(E_1)) = H^{-1}(E_1) = 0$ . Thus  $c_1(E_1)$  is effective or  $E_1 = \mathcal{O}_x$ . Since  $E_1 \in S$ , the second case does not occur. Then  $\text{Im } Z_{\sigma}(E_1) = (c_1(E_1) \cdot \omega) > 0$ . Therefore the claim holds.  $\square$

**Proposition 2.2.** Let  $B'_0$  be a connected component of  $B'$  such that  $(\beta_0, \omega_0) \in \overline{B'_0}$ . Then  $\mathcal{O}_x$  ( $x \in X$ ) is  $\sigma$ -stable for all  $\sigma \in \mathfrak{s}(B'_0)$ . In particular,  $\mathfrak{s}(\beta, \omega) = \sigma_{(\beta, \omega)}$  for  $(\beta, \omega) \in B'_0$ .

*Proof.* We set  $\sigma := \mathfrak{s}(\beta, \omega)$  ( $(\beta, \omega) \in B'_0$ ) and  $\sigma_0 := \mathfrak{s}(\beta_0, \omega_0)$ . By shrinking  $B$ , we may assume that

$$\sup_{0 \neq E \in \mathbf{D}(X)} |\phi_{\sigma'}^{\pm}(E) - \phi_{\sigma_0}^{\pm}(E)| < \frac{1}{8}$$

for  $\sigma' \in \mathfrak{s}(B)$ . Then

$$\sup_{0 \neq E \in \mathbf{D}(X)} |\phi_{\sigma}^{\pm}(E) - \phi_{\sigma'}^{\pm}(E)| < \frac{1}{4}$$

for  $\sigma, \sigma' \in \mathfrak{s}(B)$ . Since  $\mathcal{O}_x$  are  $\sigma_0$ -semi-stable,  $1 + \frac{1}{8} > \phi_{\sigma}^+(\mathcal{O}_x) \geq \phi_{\sigma}^-(\mathcal{O}_x) > 1 - \frac{1}{8}$ . For any stable factor  $E$  of  $\mathcal{O}_x$  with respect to  $\sigma$ ,  $1 + \frac{1}{8} + \frac{1}{4} > \phi_{\sigma'}^+(E) \geq \phi_{\sigma'}^-(E) > 1 - \frac{1}{8} - \frac{1}{4}$ . We set  $\mathcal{A}_{\sigma'} := \mathcal{P}_{\sigma'}((\frac{1}{2}, \frac{3}{2}]) \subset \mathbf{D}(X)$ . If  $\mathcal{O}_x$  is not  $\sigma$ -semi-stable, then let  $E_1$  be the stable factor with  $\phi_{\sigma}(E_1) = \phi_{\sigma}^+(\mathcal{O}_x) > \phi_{\sigma}(\mathcal{O}_x)$  and  $E_2 := \text{Cone}(E_1 \rightarrow \mathcal{O}_x)$ . Then we have  $E_1, E_2 \in \mathcal{A}_{\sigma'}$  for all  $\sigma' \in \mathfrak{s}(B)$  and an exact sequence

$$(2.4) \quad 0 \rightarrow E_1 \rightarrow \mathcal{O}_x \rightarrow E_2 \rightarrow 0$$

in  $\mathcal{A}_{\sigma'}$ . Since  $\mathcal{O}_x$  is  $\sigma_0$ -semi-stable with  $\phi_{\sigma_0}(\mathcal{O}_x) = 1$ ,  $\phi_{\sigma_0}(E_1) \leq \phi_{\sigma_0}(\mathcal{O}_x) = 1$ . If  $\phi_{\sigma_0}(E_1) < 1$ , then

$$\{\sigma' \in \mathfrak{s}(B) \mid \phi_{\sigma'}(E_1) < 1\}$$

is an open neighborhood of  $\sigma_0$  which does not contain  $\sigma$ , where  $\phi_{\sigma'} : \mathcal{A}_{\sigma'} \rightarrow (\frac{1}{2}, \frac{3}{2}]$ . So by shrinking  $B$ , we may assume that  $\phi_{\sigma_0}(E_1) = 1$  for all  $E_1 \in S'$ . Since  $\phi_{\sigma_0}(E_2) = \phi_{\sigma_0}(E_1) = 1$ , (2.4) is an exact sequence in  $\mathcal{A}_{(\beta_0, \omega_0)}$ . If  $\sigma \in \mathfrak{s}(B'_0)$ , then we get  $\text{Im} Z_{\sigma}(E_1) > 0 > \text{Im} Z_{\sigma}(E_2)$ , which implies  $\phi_{\sigma}(E_1) < 1 < \phi_{\sigma}(E_2)$ . Therefore  $\mathcal{O}_x$  is  $\sigma$ -stable. In particular  $\mathfrak{s}(\beta, \omega) = \sigma_{(\beta, \omega)}$  for  $(\beta, \omega) \in B'_0$  by [4, Prop. 10.3].  $\square$

In the proof of [4, Lem. 11.1], the following claim is proved.

**Lemma 2.3.** *For a bounded set  $B$  of  $\text{NS}(X)_{\mathbb{R}} \times P^+(X)_{\mathbb{R}}$ ,*

$$(2.5) \quad \Delta_B := \{u \in \Delta(X) \mid \text{rk } u > 0, Z_{(\beta, \omega)}(u) \in \mathbb{R}_{\leq 0}, (\beta, \omega) \in B\}$$

*is a finite set.*

**Proposition 2.4.** *Let  $\sigma_s$  ( $s \geq 0$ ) be a family of stability conditions such that  $Z_{\sigma_s}(\bullet) = \langle e^{\beta_s + \sqrt{-1}\omega_s}, \bullet \rangle$  and  $\mathcal{O}_x$  is  $\sigma_s$ -stable for  $s > 0$ . Assume that  $\beta_0 \in \text{NS}(X)_{\mathbb{Q}}$ ,  $\omega_0 \in \mathbb{R}_{>0}H$ ,  $H \in \text{NS}(X)$  and*

$$(2.6) \quad \{u \in \Delta(X) \mid \text{rk } u > 0, Z_{(\beta_0, \omega_0)}(u) \in \mathbb{R}_{\leq 0}\} = \emptyset.$$

*Then  $\sigma_0 = \sigma_{(\beta_0, \omega_0)} = (\mathcal{A}_{(\beta_0, \omega_0)}, Z_{(\beta_0, \omega_0)})$ .*

*Proof.* Although the claim follows from Proposition 2.2 and the covering property of  $\text{Stab}^{\dagger}(X) \rightarrow \mathcal{P}_0^+(X)$ , we shall give a more direct argument.

(Step 1) We note that  $\omega_s$  is ample for  $s > 0$  and  $\omega_0$  is nef and big.  $\mathcal{O}_x$  is  $\sigma_0$ -semi-stable. We set  $\phi_s := \phi_{\sigma_s}$ . Let  $E$  be a  $\sigma_0$ -stable object with  $0 < \phi_0 < 1$ . Then  $H^i(E) = 0$  for  $i \neq -1, 0$ , since  $E$  is a  $\sigma_s$ -stable object of  $0 < \phi_s(E) < 1$  for a small  $s > 0$  ([4, Prop. 10.3]). Let  $E$  be a  $\sigma_0$ -stable object of  $\phi_0(E) = 1$ . Assume that  $\phi_s(E) > 1$  for  $s > 0$  and set  $F := E[-1]$ . Since  $F \in \mathcal{P}_{\sigma_s}((0, 1])$  for all small  $s > 0$ ,  $H^i(F) = 0$  for  $i \neq -1, 0$  and  $H^{-1}(F)$  is torsion free ([4, Prop. 10.3]). Assume that  $H^{-1}(F) \neq 0$ , that is,  $\text{rk } H^{-1}(F) > 0$ . Since  $H^{-1}(F)[1] \in \mathcal{P}_{\sigma_s}((0, 1])$ ,  $H^{-1}(F)[1] \in \mathcal{P}_{\sigma_0}([0, 1])$ . We also have  $H^0(F) \in \mathcal{P}_{\sigma_0}([0, 1])$ . Since  $\phi_0(F) = 0$ , the exact triangle

$$H^0(F)[-1] \rightarrow H^{-1}(F)[1] \rightarrow F \rightarrow H^0(F)$$

implies  $H^{-1}(F)[1] \in \mathcal{P}_{\sigma_0}(0)$ . By the stability of  $F$ ,  $H^0(F)[-1] \in \mathcal{P}_{\sigma_0}(0)$ . Thus  $H^0(F) \in \mathcal{P}_{\sigma_0}(1)$ . Then  $H^{-1}(F) \in \mathcal{P}_{\sigma_0}(-1)$ . In particular,  $Z_{(\beta_0, \omega_0)}(H^{-1}(F)) = \text{rk } H^{-1}(F) \frac{(\omega_0^2)}{2} - a_{\beta_0}(H^{-1}(F)) < 0$ . Let  $F_1$  be a subsheaf of  $H^{-1}(F)$ . Then  $(c_1(F_1(-\beta_0)) \cdot \omega_0) > 0$  implies  $(c_1(F_1(-\beta_s)) \cdot \omega_s) > 0$  for  $1 \gg s > 0$ , which contradicts with the description of  $\mathcal{P}_{\sigma_s}((0, 1])$ . Therefore  $(c_1(F_1(-\beta_0)) \cdot \omega_0) \leq 0$ , which implies  $H^{-1}(F)$  is  $\mu$ -semi-stable with respect to  $\omega_0$ . By Lemma 2.6 below, we conclude that  $H^{-1}(F) = 0$ . In particular,  $H^i(E) = 0$  for  $i \neq -1$ . If  $\phi_s(E) \leq 1$ , then obviously  $H^i(E) = 0$  for  $i \neq -1, 0$ . Therefore  $H^i(E) = 0$  for  $i \neq -1, 0$ .

(Step 2) We take  $E \in \text{Coh}(X)$ . For  $A \in \mathcal{P}_{\sigma_0}( > 1)$ ,  $H^i(A) = 0$  for  $i \geq 0$  so that  $\text{Hom}(A, E) = 0$ . For  $B \in \mathcal{P}_{\sigma_0}( \leq -1)$ ,  $H^i(B) = 0$  for  $i \leq 0$  so that  $\text{Hom}(E, B) = 0$ . Hence  $E \in \mathcal{P}_{\sigma_0}((-1, 1])$ . For  $E \in \text{Coh}(X)$ , we have a triangle

$$D \rightarrow E \rightarrow F \rightarrow D[1]$$

such that  $D \in \mathcal{P}_{\sigma_0}((0, 1])$  and  $F \in \mathcal{P}_{\sigma_0}((-1, 0])$ . Then  $H^i(D) = 0$  for  $i \neq -1, 0$  and  $H^i(F) = 0$  for  $i \neq 0, 1$ . Taking their cohomology, we see that  $D, F \in \text{Coh}(X)$ . We set

$$(2.7) \quad \mathcal{T} := \mathcal{P}_{\sigma_0}((0, 1]) \cap \text{Coh}(X), \quad \mathcal{F} := \mathcal{P}_{\sigma_0}((-1, 0]) \cap \text{Coh}(X).$$

Then  $(\mathcal{T}, \mathcal{F})$  is a torsion pair. We show that the tilting is  $\mathcal{P}_{\sigma_0}((0, 1])$ . For  $E \in \mathcal{P}_{\sigma_0}((0, 1])$ ,  $H^{-1}(E) \in \mathcal{F}$  and  $H^0(E) \in \mathcal{T}$ . Indeed for  $F \in \mathcal{F}$ ,  $\text{Hom}(E, F) = \text{Hom}(H^{-1}(E)[2], F) = 0$  implies  $\text{Hom}(H^0(E), F) = 0$ , which shows  $H^0(E) \in \mathcal{T}$ . For  $T \in \mathcal{T}$ ,  $\text{Hom}(T[1], E) = \text{Hom}(T[1], H^0(E)[-1]) = 0$  implies  $\text{Hom}(T, H^{-1}(E)) = 0$ , which shows  $H^{-1}(E) \in \mathcal{F}$ .



(Step 3) Finally we shall prove that  $(\mathcal{T}, \mathcal{F}) = (\mathcal{T}_{(\beta_0, \omega_0)}^*, \mathcal{F}_{(\beta_0, \omega_0)}^*)$ . We note that  $Z_{(\beta, \omega)}(F) \in \mathbb{H} \cup \mathbb{R}_{<0}$  for any  $0 \neq F \in \mathcal{T}$ . Let  $E$  be an element of  $\mathcal{T}$ . Since  $F \in \mathcal{T}$  for any quotient sheaf  $F$  of  $E$ , we have  $E \in \mathcal{T}_{(\beta_0, \omega_0)}^*$ . Thus  $\mathcal{T} \subset \mathcal{T}_{(\beta_0, \omega_0)}^*$ . We also have  $\mathcal{F} \subset \mathcal{F}_{(\beta_0, \omega_0)}^*$ . Since  $(\mathcal{T}, \mathcal{F})$  is a torsion pair, the definition of  $\mathcal{T}_{(\beta_0, \omega_0)}^*, \mathcal{F}_{(\beta_0, \omega_0)}^*$  implies  $(\mathcal{T}, \mathcal{F}) = (\mathcal{T}_{(\beta_0, \omega_0)}^*, \mathcal{F}_{(\beta_0, \omega_0)}^*)$ . Hence  $(\mathcal{T}_{(\beta_0, \omega_0)}^*, \mathcal{F}_{(\beta_0, \omega_0)}^*)$  is a torsion pair of  $\text{Coh}(X)$  and  $\mathcal{P}_{\sigma_0}((0, 1]) = \mathcal{A}_{(\beta_0, \omega_0)}$  (cf. Definition 1.11).  $\square$

**Corollary 2.5.** *Assume that (2.6) holds at  $(\beta_0, \omega_0) = (\beta_0, tH)$ . Then there is a neighborhood  $B$  of  $(\beta_0, \omega_0)$  such that  $\mathfrak{s}(\beta, \omega) = \sigma_{(\beta, \omega)}$  for  $(\beta, \omega) \in B$  such that  $\omega \in \pi^*(\text{Amp}(Y)_{\mathbb{R}})$  and  $\beta \in \text{NS}(X)_{\mathbb{R}}$ .*

*Proof.* We first assume that  $\omega \in \pi^*(\text{Amp}(Y)_{\mathbb{Q}})$  and  $\beta \in \text{NS}(X)_{\mathbb{Q}}$ . By Lemma 2.3 and (2.6), we may assume that  $\Delta_B = \emptyset$ . For the family of stability conditions (2.1),  $\mathcal{O}_x$  are  $\mathfrak{s}(\beta', \omega')$ -stable if  $(\beta', \omega') \in B$  and  $\omega' \in \text{Amp}(X)_{\mathbb{R}}$ .

Then applying Proposition 2.4,  $\mathfrak{s}(\beta, \omega) = \sigma_{(\beta, \omega)}$ .

We next treat the general case. We set  $\sigma := \mathfrak{s}(\beta, \omega)$ . By the proof of Proposition 2.4, it is sufficient to show that  $H^i(E) = 0$  ( $i \neq -1, 0$ ) for all  $\sigma$ -stable object  $E$  with  $\phi_{\sigma}(E) = 1$ . Let  $U$  be a neighborhood of  $\sigma$  such that  $E$  is  $\sigma'$ -stable for all  $\sigma' \in U$ . If  $\text{rk } E \neq 0$ , then there is  $(\beta', \omega') \in \text{NS}(X)_{\mathbb{Q}} \times \pi^*(\text{Amp}(Y)_{\mathbb{Q}})$  such that  $\sigma' := \mathfrak{s}(\beta', \omega') \in U$  and  $\phi_{\mathfrak{s}(\beta', \omega')}(E) < 1$ . Hence  $H^i(E) = 0$  for  $i \neq -1, 0$ . Assume that  $\text{rk } E = 0$ . If  $c_1(E) \notin \pi^*(\text{Amp}(Y))^{\perp}$ , then we can take  $(\beta', \omega')$  such that  $\phi_{\sigma'}(E) < 1$ , which implies  $H^i(E) = 0$  for  $i \neq -1, 0$ . If  $c_1(E) \in \pi^*(\text{Amp}(Y))^{\perp}$ , then  $\phi_{\sigma'}(E) = 1$ , which also implies  $H^i(E) = 0$  for  $i \neq -1, 0$ .  $\square$

**Lemma 2.6.** *Assume that  $\beta_0$  and  $\omega_0$  are rational, and satisfy (2.6). Then there is no  $\mu$ -semi-stable sheaf  $E$  of  $\text{rk } E > 0$  with respect to  $\omega_0$  such that  $Z_{(\beta_0, \omega_0)}(E) \in \mathbb{R}_{<0}$ .*

*Proof.* Let  $E$  be a  $\mu$ -semi-stable sheaf with  $\deg_{\beta_0}(E) = 0$ . Let  $\mathfrak{C}$  be a category of perverse coherent sheaves associated to  $\beta_0$ . Then we have a decomposition

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$$

such that  $E_1 \in \mathfrak{C} \cap \text{Coh}(X)$  and  $E_2 \in \mathfrak{C}[-1] \cap \text{Coh}(X)$ . Then  $a_{\beta_0}(E_2) = \chi_{\beta_0}(E_2) \leq 0$ , and hence  $a_{\beta_0}(E) \leq a_{\beta_0}(E_1)$ . In particular,  $Z_{(\beta_0, \omega_0)}(E) \geq Z_{(\beta_0, \omega_0)}(E_1) \in \mathbb{R}_{\leq 0}$ . Since  $E$  is  $\mu$ -semi-stable with respect to  $\omega_0$ ,  $E_1$  is a  $\mu$ -semi-stable perverse coherent sheaf. In  $\mathfrak{C}$ ,  $E_1$  is generated by  $\beta$ -twisted stable torsion free objects  $F$  with  $\deg_{\beta_0}(F) = 0$ . If  $\langle v(F)^2 \rangle \geq 0$ , then  $a_{\beta_0}(F) \leq 0$ , and hence  $Z_{(\beta, \omega_0)}(F) \in \mathbb{R}_{>0}$ . If  $v(F) \in \Delta(X)$ , then our assumption implies  $Z_{(\beta, \omega_0)}(F) \in \mathbb{R}_{>0}$ . Therefore  $Z_{(\beta, \omega)}(E_1) \in \mathbb{R}_{>0}$ .  $\square$

**Proposition 2.7.** *Let  $\sigma_s$  ( $0 \geq s \gg -1$ ) be a family of stability conditions such that  $Z_{\sigma_s}(\bullet) = \langle e^{(\beta+sH)+iH}, \bullet \rangle$ . Assume that irreducible objects  $A \in \mathfrak{C}$  are  $\sigma_s$ -semi-stable with  $\phi_s(A) = 1$  and  $\sigma_s$ -stable for  $s < 0$ . Then  $\sigma_0 = (\mathcal{A}_{(\beta, H)}, Z_{(\beta, H)})$ .*

*Proof.* We first prove that  ${}^p H^i(E) = 0$  ( $i \neq -1, 0$ ) for all  $E \in \mathcal{P}_{\sigma_0}((0, 1])$ . Let  $E$  be a  $\sigma_0$ -stable object of  $\mathcal{P}_{\sigma_0}(\phi)$  with  $0 < \phi < 1$ . Let  $A$  be an irreducible object of  $\mathfrak{C}$ . Then  $\text{Hom}(E[i], A) = 0$  for  $i > 0$  by  $\phi_{\sigma_0}(E[i]) = \phi + i > 1$ . Hence  ${}^p H^i(E) = 0$  for  $i > 0$ .  $\text{Hom}(E[i], A) = \text{Hom}(A, E[i+2])^{\vee} = 0$  for  $i \leq -2$  by  $\phi_{\sigma_0}(E[i+2]) = \phi + i + 2 < 1$ . Hence  ${}^p H^i(E) = 0$  for  $i \neq -1, 0$ .

Let  $E$  be a  $\sigma_0$ -stable object of  $\mathcal{P}_{\sigma_0}(\phi)$  with  $\phi = 1$ . Assume that  $\phi_{\sigma_s}(E) > 1$  for  $s < 0$ . We set  $F := E[-1]$ . Then  ${}^p H^i(F) = 0$  for  $i \neq -1, 0$ . We set

$$(2.8) \quad v_i := v({}^p H^{-i}(E)) = e^{\beta}(r_i + d_i H + D_i + a_i \varrho_X), \quad D_i \in H^{\perp}, \quad (i = 0, 1).$$

Since  $\phi_{\sigma_0}(F) = 0$ ,  $d_0 - d_1 = 0$ . By the description of  $\sigma_s$  ( $s < 0$ ),  $d_0 - r_0 s \geq 0$  and  $d_1 - r_1 s \leq 0$  for  $0 > s \gg -1$ . Hence  $d_0 \geq 0$  and  $d_1 \leq 0$ , which implies  $d_0 = d_1 = 0$ . Then  $d_1 - r_1 s \leq 0$  and  $s < 0$  implies  $r_1 = 0$ . Hence  ${}^p H^{-1}(F)$  is a torsion object of  $\mathfrak{C}$ . By the description of  $\mathcal{P}_{\sigma_s}((0, 1])$ ,  ${}^p H^{-1}(F) = 0$ . If  $\phi_{\sigma_s}(E) \leq 1$  for  $s < 0$ , then we also have  ${}^p H^i(E) = 0$  for  $i \neq -1, 0$ .

Then it is easy to see that  $\mathcal{T} := \mathcal{P}_{\sigma_0}((0, 1]) \cap \mathfrak{C}$  and  $\mathcal{F} := \mathcal{P}_{\sigma_0}((-1, 0]) \cap \mathfrak{C}$  is a torsion pair of  $\mathfrak{C}$ ,  $\mathcal{P}_{\sigma_0}((0, 1])$  is the tilting and  $(\mathcal{T}, \mathcal{F}) = (\mathcal{T}_{(\beta, H)}, \mathcal{F}_{(\beta, H)})$ .  $\square$

**2.2. A family of stability conditions parametrized by a half plane.** We consider stability conditions

$$(2.9) \quad P_{\gamma, H} := \{\sigma_{(\gamma+sH, tH)} \mid s \in \mathbb{R}, t \in \mathbb{R}_{>0}, Z_{(\gamma+sH, tH)}(u) \neq 0 \ (u \in \Delta(X))\}.$$

To be more precise,  $\sigma_{(\gamma+sH, tH)}$  is well-defined on a simply connected open subset such that  $Z_{(\gamma+sH, tH)}(u) \notin \mathbb{R}_{\leq 0}$  for all  $u \in \Delta(X)$  with  $\text{rk } u > 0$ . We shall regard  $\sigma_{(\gamma+sH, tH)}$  as a limit as in Proposition 2.7 if  $Z_{(\gamma+sH, tH)}(u) \in \mathbb{R}_{<0}$  for a Mukai vector  $u \in \Delta(X)$  with  $\text{rk } u > 0$ .

For a Mukai vector  $v$ , let  $U_v$  be a chamber in  $P_{\gamma, H}$ . For a wall  $W$  in  $\text{Stab}(X)$ , we have

- (1)  $P_{\gamma, H} \subset W$  or
- (2)  $W$  and  $P_{\gamma, H}$  intersect properly and  $W \cap P_{\gamma, H}$  is a wall for  $v$ .

Hence if  $\mathcal{C}_v$  is a chamber in  $\text{Stab}(X)$  with  $\overline{\mathcal{C}_v} \cap U_v \neq \emptyset$ , then  $U_v \subset \overline{\mathcal{C}_v}$ . Let

$$v_1 = e^\gamma(r_1 + d_1H + D_1 + a_1\varrho_X), \quad D_1 \in H^\perp$$

be a Mukai vector which defines a wall  $W$  for  $v = e^\gamma(r + dH + D + a\varrho_X)$  ( $D \in H^\perp$ ). Then

$$P_{\gamma,H} \subset W \iff (r_1, d_1, a_1) \in \mathbb{Q}(r, d, a).$$

For  $v = \varrho_X$ , there is a chamber

$$\mathcal{C}_\varrho := \{(\beta, \omega) \mid (C \cdot \omega) > 0 \text{ for any exceptional curve } C\} = \{(\beta, \omega) \mid \omega \in \text{Amp}(X)_\mathbb{R}\}$$

in  $\text{NS}(X)_\mathbb{R} \times P^+(X)_\mathbb{R}$ . For  $v_0 := r_0e^\gamma$ , let  $U_{v_0}$  be a chamber in  $P_{\gamma,H}$ . We take  $(\beta_0, \omega_0) := (\gamma + s_0H, t_0H) \in U_{v_0}$ . We take  $(\beta, \omega)$  in an adjacent chamber  $\mathcal{C}$  of  $U_{v_0}$ , and let  $\mathcal{E} \in \mathbf{D}(X \times X')$  be a universal family (as a twisted object in general) of  $\sigma_{(\beta, \omega)}$ -stable objects with the Mukai vector  $v_0$ , where  $X'$  is the coarse moduli scheme of  $\sigma_{(\beta, \omega)}$ -stable objects. Let

$$\Phi := \Phi_{X \rightarrow X'}^{\mathcal{E}^\vee[2]} : \mathbf{D}(X) \rightarrow \mathbf{D}(X')$$

be a Fourier-Mukai transform defined by

$$(2.10) \quad \Phi(E) := \mathbf{R}p_{X'*}(p_X^*(E) \otimes \mathcal{E}^\vee[2]),$$

where  $p_X, p_{X'}$  are projections from  $X \times X'$  to  $X$  and  $X'$  respectively.  $\Phi$  induces an isomorphism

$$(2.11) \quad \begin{array}{ccc} \Phi : \text{NS}(X)_\mathbb{R} \times P^+(X)_\mathbb{R} & \rightarrow & \text{NS}(X')_\mathbb{R} \times P^+(X')_\mathbb{R} \\ (\beta, \omega) & \mapsto & (\beta', \omega'), \end{array}$$

where  $(\beta', \omega')$  is defined by

$$(2.12) \quad e^{\beta' + i\omega'} := \frac{\Phi(e^{\beta + i\omega})}{-\langle e^{\beta + i\omega}, v_0 \rangle}.$$

For  $(\beta, \omega) \in \mathcal{C}$ ,  $\omega'$  is ample and  $\Phi(\sigma_{(\beta, \omega)}) \equiv \sigma_{(\beta', \omega')} \pmod{\widetilde{GL}_2^+(\mathbb{R})}$ . For  $(\beta, \omega) = (\gamma + sH, tH)$ , we set  $(\beta', \omega') := (\gamma' + s'H', t'H')$ , where  $\Phi(\varrho_X) = r_0e^{\gamma'}$  and  $\Phi(e^\gamma H) = -e^{\gamma'}H'$ . Then  $\Phi(U_{v_0})$  is a chamber in  $P_{\gamma', H'}$ . Since  $\Phi(\overline{\mathcal{C}}) \supset \Phi(U_{v_0}) \ni (\beta'_0, \omega'_0)$ , there is a category of perverse coherent sheaves  $\mathfrak{C}'$  associated to a contraction  $\pi' : X' \rightarrow Y'$  by  $H'$ , and  $\sigma_{(\beta', \omega')}$  ( $(\beta, \omega) \in U_{v_0}$ ) is the stability condition in subsection 1.2 (Proposition 2.4).

We set  $L := (e^\gamma)^\perp \cap (He^\gamma)^\perp \cap v(K(X))$ . Then  $L$  is negative semi-definite. Let

$$(2.13) \quad r_0e^\gamma = \sum_i n_i u_i, \quad (n_i \in \mathbb{Z}_{>0}, u_i \in L)$$

be a decomposition of  $r_0e^\gamma$  in  $L$  such that

- (i)  $u_i$  are indecomposable Mukai vectors with  $\langle u_i^2 \rangle = -2$ ,
- (ii)  $\text{rk } u_i / r_0 > 0$  and  $\sum_i n_i \text{rk } u_i / r_0 = 1$ ,

where  $u_i$  is indecomposable, if there is no decomposition  $u_i = \sum_j n_{ij} u_{ij}$  ( $n_{ij} \in \mathbb{Z}_{>0}, u_{ij} \in L$ ) such that  $\langle u_{ij}^2 \rangle = -2$  and  $\text{rk } u_{ij} / r_0 > 0$ . By [16, Lem. A.1.1], the sublattice  $\oplus_i \mathbb{Z}u_i$  is of type  $\tilde{A}, \tilde{D}, \tilde{E}$ .

Then there are  $\sigma_{(\beta_0, \omega_0)}$ -semi-stable objects  $E_i$  with  $v(E_i) = u_i$ . Since  $u_i$  are indecomposable,  $E_i$  are  $\sigma_{(\beta_0, \omega_0)}$ -stable.

**Lemma 2.8.** *Let  $E$  be a  $\sigma_{(\beta_0, \omega_0)}$ -stable object with  $\phi_{(\beta_0, \omega_0)}(E) = \phi_{(\beta_0, \omega_0)}(r_0e^\gamma)$  and  $v(E) \neq r_0e^\gamma$ . Assume that  $E$  satisfies  $\langle v(E), e^\gamma \rangle = 0$ . Then  $\langle v(E)^2 \rangle = -2$  and  $E$  is a stable factor of  $\mathcal{E}_{\{x'\} \times X}$ .*

*Proof.*  $\Phi(E)$  is a  $\sigma_{(\beta'_0, \omega'_0)}$ -stable object with  $\phi_{(\beta'_0, \omega'_0)}(\Phi(E)) = 1$ . In particular,  $\Phi(E) \in \mathcal{A}_{(\beta'_0, \omega'_0)}$ . Since  $\text{rk } \Phi(E) = 0$  and irreducible,  ${}^pH^i(\Phi(E)) = 0$  for  $i \neq -1$  or  $i \neq 0$ . Since  ${}^pH^{-1}(\Phi(E))$  is torsion free,  $\Phi(E) \in \mathfrak{C}'$ . Thus  $\Phi(E)$  is a 0-dimensional object of  $\mathfrak{C}'$ . We note that  $F \in \mathcal{A}_{(\beta'_0, \omega'_0)}$  is an irreducible object with  $\text{rk } F = 0$  if and only if  $F$  is an irreducible object of  $\mathfrak{C}'$ . Hence  $\langle v(E)^2 \rangle = \langle v(\Phi(E))^2 \rangle = -2$ . Since every irreducible object of  $\mathfrak{C}'$  is a stable factor of  $\mathcal{O}_{x'}$  by [16, Lem. 1.1.21],  $E$  is a stable factor of  $\mathcal{E}_{\{x'\} \times X}$ .  $\square$

An object  $E \in \mathfrak{C}'$  with  $v(E) = \varrho_{X'}$  is  $\nu$ -stable in the sense of [16, sect. 2.2], if  $-(\nu, c_1(F_1)) \leq 0$  for all subobject  $F_1$  of  $E$ . Hence it is the same as  $\sigma_{(\beta', \omega')}$ -semi-stability, where  $(\beta', \omega') = (\gamma' + s'H' + \mu, t'H' + \nu)$  is sufficiently close to  $(\beta'_0, \omega'_0)$ . If  $\nu$  is relatively ample with respect to  $\pi'$ , then  $\mathcal{O}_{x'}$  is  $\nu$ -stable.

Since  $\mathcal{M}_{(\beta_0, \omega_0)}(r_0e^\gamma) \cong \mathcal{M}_{(\beta'_0, \omega'_0)}(\varrho_{X'})$ , [16, Prop. 2.2.8] implies  $\mathcal{M}_{(\beta_0, \omega_0)}(r_0e^\gamma)$  is irreducible. Moreover the  $S$ -equivalence classes of properly  $\sigma_{(\beta_0, \omega_0)}$ -semi-stable objects are  $\oplus_i E_i^{\oplus n_i}$ , where  $v_0 = \sum_i n_i v(E_i)$  is a decomposition of (2.13).

**Theorem 2.9.** *Let  $U_{v_0}$  be a chamber for  $v_0 = r_0e^\gamma$  in  $P_{\gamma,H}$  and  $(\beta_0, \omega_0) \in U_{v_0}$ . Assume that  $\text{char}(k) = 0$  or  $M_{(\beta, \omega)}(r_0e^\gamma)$  is a fine moduli space for  $(\beta, \omega)$  in an adjacent chamber  $\mathcal{C}$  of  $U_{v_0}$ . Then  $M_{(\beta_0, \omega_0)}(r_0e^\gamma)$  is isomorphic to a normal K3 surface  $Y'$  which is obtained as a contraction  $\pi' : X' \rightarrow Y'$  by the nef and big divisor  $\omega'_0$ .*

*Proof.* If  $\text{char}(k) = 0$ , then [16, Prop. 2.2.11] implies  $M_{(\beta'_0, \omega'_0)}(\varrho_{X'}) = (X')^0$  is normal. Hence the claim holds. If  $M_{(\beta_0, \omega_0)}(r_0 e^\gamma)$  is a fine moduli space, then  $M_{(\beta'_0, \omega'_0)}(\varrho_{X'}) = (X')^0$  is the moduli of untwisted 0-dimensional objects of Mukai vector  $\varrho_{X'}$ , where  $(X')^0$  is the moduli of 0-semi-stable objects in [16, Def. 2.2.1]. In this case, there is an autoequivalence  $\Phi$  of  $\mathbf{D}(X')$  such that  $\Phi(\mathfrak{C}) = {}^{-1}\text{Per}(X'/Y')$  by [16, Prop. 2.3.27]. Applying [16, Rem. 2.2.13], we see that  $(X')^0 \cong Y'$ .  $\square$

### 3. STABILITY CONDITIONS ON AN ENRIQUES SURFACE

**3.1. 2-dimensional moduli spaces.** The space of stability conditions on an Enriques surface was studied in [8] by comparing the stability conditions on the covering K3 surface. In this section, we shall explain some of the results. For this purpose, we prepare some notations. Let  $X$  be a classical Enriques surface over  $k$ , that is  $K_X \neq 0$ . As in the case of K3 surfaces, we introduce the following definition.

**Definition 3.1.** (1) For  $E \in \mathbf{D}(X)$ ,

$$v(E) = \text{ch}(E)\sqrt{\text{td}_X} = \text{rk } E + c_1(E) + \left( \text{ch}_2(E) + \frac{\text{rk } E}{2} \varrho_X \right) \in H^*(X, \mathbb{Q})$$

is the Mukai vector of  $E$ . Let  $(v(K(X)), \langle \cdot, \cdot \rangle)$  be the Mukai lattice of  $X$ .

- (2) Let  $\Delta(X)$  be the subset of  $v(K(X))$  consisting of  $u = (r, \xi, \frac{b}{2})$  such that (i)  $\langle u^2 \rangle = -1$  or (ii)  $\langle u^2 \rangle = -2$  and  $\xi \equiv D \pmod{2}$ , where  $D$  is a nodal cycle.

As in [4], let  $\mathcal{P}(X)$  be the subset of  $v(K(X))_{\mathbb{C}}$  consisting of  $\bar{U}$  such that  $\text{Re } \bar{U}$  and  $\text{Im } \bar{U}$  span a positive definite 2-plane in  $v(K(X))_{\mathbb{R}}$ . We shall regard  $\mathcal{P}(X)$  as a subset of  $\text{Hom}(v(K(X)), \mathbb{C})$  by  $\langle \bar{U}, \bullet \rangle$ . Let  $\mathcal{P}^+(X)$  be the connected component of  $\mathcal{P}(X)$  containing  $e^{\beta+i\omega}$ , where  $\omega$  is ample. We set

$$\mathcal{P}_0^+(X) := \mathcal{P}^+(X) \setminus \bigcup_{u \in \Delta(X)} u^\perp.$$

Then for the connected component  $\text{Stab}^\dagger(X)$  containing geometric stability conditions,

$$(3.1) \quad \begin{array}{ccc} \text{Stab}^\dagger(X) & \rightarrow & \mathcal{P}_0^+(X) \\ \sigma & \mapsto & Z_\sigma \end{array}$$

is a covering map with the group of deck transformations  $\text{Aut}^0(\mathbf{D}(X))/\langle \otimes K_X \rangle$  [8, Cor. 3.8] at least if  $k = \mathbb{C}$ . By the same argument of Bridgeland [4, Prop. 13.2], we have

$$(3.2) \quad \text{Stab}^\dagger(X) = \bigcup_{\Phi \in \mathbf{T}} \Phi(\overline{U(X)})$$

where  $\mathbf{T} \subset \text{Aut}(\mathbf{D}(X))$  is the subgroup of autoequivalences generated by twist functors  $T_A^2$  and  $T_{\mathcal{O}_C(k)}$  (see Definition 4.4), where  $A$  is a spherical object or an exceptional object, and  $C$  is a  $(-2)$  curve on  $X$ . Thus  $\Phi(U(X))$  is the chamber of  $\varrho_X$  and we have a fine moduli space for every chamber (see [5] for the corresponding result on a K3 surface).

**Theorem 3.2.** (1) Let  $v_0 = (r, \xi, \frac{s}{2})$  be a primitive and isotropic Mukai vector such that  $\gcd(r, \xi, s) = 2$ . Let  $\mathcal{C}$  be a chamber with respect to  $v_0$ . Then  $M_{(\beta, \omega)}(v_0)$  ( $(\beta, \omega) \in \mathcal{C}$ ) is a fine moduli space and  $M_{(\beta, \omega)}(v_0) \cong X$ .

- (2) For  $\eta/p$  such that  $\eta \in \text{NS}(X)$  and  $p$  is an odd integer, a primitive element  $v_0 \in \mathbb{Q}e^{\eta/p} \cap v(K(X))$  satisfies the assumption.

*Proof.* (1) By (3.2),  $M_\sigma(\varrho_X) \cong X$  if  $\sigma$  belongs to a chamber. We treat the general case, by using a special kind of Fourier-Mukai transforms. Let  $\mathbf{G}$  be the subgroup of  $\text{Aut}(\mathbf{D}(X))$  generated by the following autoequivalences:

- (i) A twist functor  $T_{\mathcal{O}_X}$ .  $T_{\mathcal{O}_X}$  is a Fourier-Mukai transform  $\Phi_{X \rightarrow X}^{\mathcal{E}[1]}$ , where  $\mathcal{E}_{|\{x\} \times X}$  is a stable sheaf with Mukai vector  $w_0 := v(\mathcal{O}_X \oplus \mathcal{O}_X(K_X)) - \varrho_X$ .  
(ii) For  $D \in \text{NS}(X)$ ,

$$(3.3) \quad \begin{array}{ccc} \mathcal{L}_D : \mathbf{D}(X) & \rightarrow & \mathbf{D}(X) \\ E & \mapsto & E(D). \end{array}$$

- (iii) The shift functor:

$$(3.4) \quad \begin{array}{ccc} [1] : \mathbf{D}(X) & \rightarrow & \mathbf{D}(X) \\ E & \mapsto & E[1]. \end{array}$$

The equivalence  $\Phi_{X \rightarrow X}^{\mathcal{E}} : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$  in (i) induces an isomorphism  $\text{Stab}^\dagger(X) \rightarrow \text{Stab}^\dagger(X)$ , and hence isomorphisms  $\mathcal{M}_\sigma(v) \rightarrow \mathcal{M}_{\Phi_{X \rightarrow X}^{\mathcal{E}}(\sigma)}(v')$ , where  $v' := \Phi_{X \rightarrow X}^{\mathcal{E}}(v)$ . Indeed since  $M_H(w_0) = M_{(\beta, \omega)}(w_0)$  for some  $(\beta, \omega) \in \text{NS}(X)_{\mathbb{Q}} \times \text{Amp}(X)_{\mathbb{Q}}$  with  $\sigma_{(\beta, \omega)} \in U(X)$ , there is  $\sigma' \in U(X) \subset \text{Stab}^\dagger(X)$  such that  $\Phi_{X \rightarrow X}^{\mathcal{E}}(\sigma') = \sigma_{(\beta, \omega)} \in U(X) \subset \text{Stab}^\dagger(X)$ , which implies the connected component  $\text{Stab}^\dagger(X)$  is preserved



under  $\Phi_{X \rightarrow X}^\varepsilon$ . Obviously  $\mathcal{L}_D$  and [1] also preserve the stability. Since there is an autoequivalence  $\Phi \in \mathbf{G}$  such that  $\Phi(v_0) = \varrho_X$  (cf. [19]), we get  $M_\sigma(v_0) \cong M_{\Phi(\sigma)}(\varrho_X) \cong X$ .

(2) We set  $v_0 := lp e^{\eta/p} = (lp, l\eta, \frac{s}{2})$ . Then we see that  $lp$  is even. Since  $p$  is odd,  $l$  is even, which implies  $\gcd(lp, l\eta, s) = 2$  by  $lp \equiv s \pmod{2}$ .  $\square$

*Remark 3.3.* The essential part of (1) is the existence of a Fourier-Mukai transform associated to  $v_0$ , which was first proved by Nuer [12].

*Remark 3.4.* Assume that  $k$  is not algebraically closed. If all divisors on  $X$  are defined over  $k$ , then  $M_H^\beta(v)$  has a universal family if  $v$  is primitive, by the unimodularity of Mukai lattice. If  $\langle v^2 \rangle = -1, -2$ , then  $M_H(v)$  is a reduced one point. Hence there is a  $\beta$ -stable object  $E$  with  $v(E) = v$ . Then  $\Phi \in \mathbf{T}$  are defined over  $k$ .

**3.2. Gieseker chambers on an Enriques surface.** The results in section 2 hold for the case of Enriques surfaces. In [21], we studied Gieseker chambers in a 2-dimensional subspace of  $\text{Stab}^\dagger(X)$  for a primitive and isotropic Mukai vector  $v_0$  on a K3 surface. In this section, we present a similar result for stability conditions associated to a category of perverse coherent sheaves on a K3 surface and also an Enriques surface  $X$ . Let  $H$  be a nef and big divisor which defines a contraction  $\pi : X \rightarrow Y$  of  $(-2)$ -curves  $C \in H^\perp$ . Let  $\mathfrak{E}$  be a category of perverse coherent sheaves with a local projective generator  $G$ . We set  $\beta := c_1(G)/\text{rk } G$ . Let  $v_0 := r_0 e^\gamma$  be a primitive and isotropic Mukai vector such that  $X' := M_H^\beta(v_0)$  is a smooth surface, that is,  $v_0 = (r_0, \xi, \frac{b}{2})$  with  $\gcd(r_0, \xi, b) = 2$ . We set  $\beta = \gamma + sH + \mu$  ( $\mu \in H^\perp$ ) and assume that  $\mu$  is sufficiently close to 0.

**Definition 3.5.** For a Mukai vector  $v$ , we set  $\epsilon = 1, 2$  according as  $\text{rk } v$  is odd or even. For a stable object  $E$ ,  $\langle v(E)^2 \rangle \geq -\epsilon$ .

Let  $\sigma_{(\gamma+sH, tH)}$  be a family of stability condition associated to  $\mathfrak{E}$ . We set

$$\delta := \frac{1}{(H^2)} \min\{(D, H) > 0 \mid D \in \text{NS}(X)\}.$$

**Definition 3.6.** (1) Let  $\mathfrak{E}_\epsilon$  ( $\epsilon = 1, 2$ ) be the set of Mukai vectors

$$v_1 = e^\gamma(r_1 + d_1 H + D_1 + a_1 \varrho_X), \quad D_1 \in H^\perp$$

such that  $v_1 \in \Delta(X)$ ,  $\langle v_1^2 \rangle = -\epsilon$  and  $r_1 > 0, d_1 < 0, a_1 > 0$ .

(2) For  $v_1 \in \mathfrak{E} = \mathfrak{E}_1 \cup \mathfrak{E}_2$ , we set

$$f_{v_1}(s) := \begin{cases} \sqrt{\frac{2}{(H^2)} \frac{a_1}{d_1} s - s^2}, & s \in [\frac{2}{(H^2)} \frac{a_1}{d_1}, \frac{d_1}{r_1}], \\ 0 & \text{otherwise.} \end{cases}$$

(3) We set

$$f(s) := \max_{v_1 \in \mathfrak{E}} f_{v_1}(s).$$

The following result characterize the Gieseker chamber for  $v_0$ .

**Proposition 3.7** ([21, Prop. 1.11]). *Assume that  $s$  is rational.*

- (1) *If  $t > f(s)$ , then  $M_{(\gamma+sH, tH)}(r_0 e^\gamma) = M_H^\gamma(r_0 e^\gamma)$ .*
- (2) *If  $t < f(s)$ , then all  $E \in M_H^\gamma(r_0 e^\gamma)$  are not  $\sigma_{(\gamma+sH, tH)}$ -semi-stable.*

In the same way as in [21], we get the following results.

**Lemma 3.8** ([21, Lem. 1.13]).

$$f_{v_1}(s) \leq \max \left\{ \sqrt{\frac{\epsilon}{(H^2)}}, \sqrt{-\frac{4r_0}{(H^2)\delta} s - s^2} \right\}.$$

**Proposition 3.9** ([21, Prop. 1.14]). *Assume that*

$$(3.5) \quad |s| \leq \min \left\{ \frac{1}{r_0} \sqrt{\frac{1}{\epsilon(H^2)}}, \frac{\delta}{\epsilon r_0^2} \right\}.$$

*Then*

$$(3.6) \quad f_{v_1}(s) \leq \frac{1}{r_0} \sqrt{\frac{1}{\epsilon(H^2)}} - \sqrt{\frac{1}{r_0^2} \frac{1}{\epsilon(H^2)} - s^2}$$

*for all  $v_1 \in \mathfrak{E}_\epsilon$ . In particular if*

$$(3.7) \quad |s| \leq \min \left\{ \frac{1}{r_0} \sqrt{\frac{1}{2(H^2)}}, \frac{\delta}{2r_0^2} \right\},$$

then

$$(3.8) \quad f(s) \leq \frac{1}{r_0} \sqrt{\frac{1}{2(H^2)}} - \sqrt{\frac{1}{r_0^2} \frac{1}{2(H^2)}} - s^2.$$

**3.3. A bound on the Gieseker chamber.** We set

$$s_0 := \min \left\{ \frac{1}{r_0} \sqrt{\frac{1}{2(H^2)}}, \frac{\delta}{2r_0^2} \right\}.$$

By the description of  $\mathfrak{E}$  and Corollary 2.5, we get the following.

**Proposition 3.10.** *Assume that*

$$(3.9) \quad 0 < |s| < s_0, \quad t > \frac{1}{r_0} \sqrt{\frac{1}{2(H^2)}} - \sqrt{\frac{1}{2r_0^2(H^2)}} - s^2.$$

Then  $\mathcal{O}_x$  ( $x \in X$ ) is  $\sigma_{(\gamma+sH, tH)}$ -semi-stable such that all stable factors are irreducible objects of  $\mathfrak{E}$ .

Let  $v = e^\beta(r + D + a\varrho_X)$ , ( $D \in H^\perp$ ) be a Mukai vector. We set

$$p_0 := \frac{\frac{2}{(H^2)} + s_0^2 + \frac{\langle v^2 \rangle - (D^2)}{r^2(H^2)}}{2s_0}.$$

Let  $V_v(X)$  be the open subset defined by

$$(3.10) \quad \begin{aligned} s &\neq 0, \quad t^2 + (|s| - p_0)^2 > p_0^2 - \frac{\langle v^2 \rangle - (D^2)}{r^2(H^2)}, \\ t &\geq \frac{1}{r_0} \sqrt{\frac{1}{2(H^2)}} - \sqrt{\frac{1}{2r_0^2(H^2)}} - s^2. \end{aligned}$$

If  $t > \sqrt{\frac{2}{(H^2)}}$ , then  $Z_{(\gamma+sH, tH)}(u) \notin \mathbb{R}_{\leq 0}$  for  $u \in \Delta(X)$  with  $\text{rk } u > 0$ .

Assume that  $(\gamma+s_1H, t_1H)$  belongs to a Gieseker chamber for a Mukai vector  $v$ , that is,  $\mathcal{M}_{(\gamma+s_1H, t_1H)}(v) = \mathcal{M}_H^\gamma(v)$ . We set

$$p := \min \left\{ \frac{t_1^2 + s_1^2 + \frac{\langle v^2 \rangle - (D^2)}{r^2(H^2)}}{2s_1}, -p_0 \right\}.$$

Assume that  $\mathcal{M}_{(\gamma+s'_1H, t'_1H)}(v) = \{E \mid E^\vee \in \mathcal{M}_H^{-\gamma}(v^\vee)\}$ . We set

$$p' := \max \left\{ \frac{t_1'^2 + s_1'^2 + \frac{\langle v^2 \rangle - (D^2)}{r^2(H^2)}}{2s_1'}, p_0 \right\}.$$

By [9, Cor. 3.2.10] and [20, Cor. 3.6], we get the following.

**Proposition 3.11.** *Let  $X$  be a K3 surface or an Enriques surface. Let  $v = e^\beta(r + D + a\varrho_X)$ , ( $D \in H^\perp$ ) be a Mukai vector.*

(1) *Assume that  $(s, t)$  satisfies*

$$s < 0, \quad t^2 + (s - p)^2 > p^2 - \frac{\langle v^2 \rangle - (D^2)}{r^2(H^2)}, \quad t \geq \frac{1}{r_0} \sqrt{\frac{1}{2(H^2)}} - \sqrt{\frac{1}{2r_0^2(H^2)}} - s^2.$$

*Then  $\mathcal{M}_{(\gamma+sH, tH)}(v)^{ss} = \mathcal{M}_H^\gamma(v)^{ss}$ .*

(2) *Assume that  $(s, t)$  satisfies*

$$s > 0, \quad t^2 + (s - p')^2 > p'^2 - \frac{\langle v^2 \rangle - (D^2)}{r^2(H^2)}, \quad t \geq \frac{1}{r_0} \sqrt{\frac{1}{2(H^2)}}.$$

*Then  $\mathcal{M}_{(\gamma+sH, tH)}(v) = \{E \mid E^\vee \in \mathcal{M}_H^{-\gamma}(v^\vee)\}$ .*

**3.4. An isomorphism by a Fourier-Mukai transform.** We consider a family of stability conditions  $P_{\gamma,H}$  in (2.9). Let  $v_0 := r_0 e^\gamma$  be a primitive isotropic Mukai vector such that  $\dim M_{(\beta,\omega)}(v_0) = 2$  for a general  $(\beta, \omega)$ . Let  $C_v \neq \emptyset$  be a semi-circle defined by  $\mathbb{R}_{>0} Z_{(\gamma+sH,tH)}(r_0 e^\gamma) = \mathbb{R}_{>0} Z_{(\gamma+sH,tH)}(v)$ , where  $v$  is a Mukai vector with  $\langle v^2 \rangle \geq 0$ . Since  $\langle v^2 \rangle, \langle v_0^2 \rangle \geq 0$ , [10, (5.10)] implies  $\langle v, v_0 \rangle > 0$ .

Let  $(\gamma + s_0 H, t_0 H)$  be a point of  $C_v$  and  $U$  a neighborhood of  $(\gamma + s_0 H, t_0 H)$ . Let  $U_\pm$  be the connected components of  $U \setminus C_v$  such that

$$U_\pm \subset \{(\gamma + sH, tH) \mid \pm(\phi_{(\gamma+sH,tH)}(v) - \phi_{(\gamma+sH,tH)}(v_0)) > 0\}.$$

For chambers  $\mathcal{C}_\pm$  with  $U_\pm \subset \overline{\mathcal{C}_\pm}$ , we consider moduli schemes  $X' := M_{(\beta,\omega)}(r_0 e^\gamma)$  ( $(\beta, \omega) \in \mathcal{C}_\pm$ ). Then  $X'$  is a K3 surface or an Enriques surface. Let  $\mathcal{E}_\pm \in \mathbf{D}(X \times X')$  be universal families (as twisted objects). Let

$$\Phi_\pm := \Phi_{X \rightarrow X'}^{\mathcal{E}_\pm^\vee[2]} : \mathbf{D}(X) \rightarrow \mathbf{D}(X')$$

be a Fourier-Mukai transform in (2.10). We use the notation in subsection 2.2. Then  $\Phi_\pm(C_v)$  is the line defined by  $s' = s'_0$ , where  $\Phi(\gamma + s_0 H, t_0 H) = (\gamma' + s'_0 H', t'_0 H')$ . We set  $U'_\pm := \Phi_\pm(U_\pm)$ . By shrinking  $U$ , we may assume that there is no  $(\gamma' + s' H', t' H') \in U'_\pm$  such that  $Z_{(\gamma'+s'H',t'H')}(u) \notin \mathbb{R}_{\leq 0}$  for a Mukai vector  $u \in \Delta(X)$  with  $\text{rk } u > 0$ . Then  $\sigma_{(\gamma'+s'H',t'H')}$  is the stability condition associated to a category of perverse coherent sheaves if  $(\gamma' + s' H', t' H') \in U'_\pm$ .

For  $E \in \mathcal{M}_{(\gamma+s_0 H, t_0 H)}(v)$ , we set  $F := \Phi_{X \rightarrow X'}^{\mathcal{E}_\pm^\vee[2]}(E)$ . Then  $\Phi_\pm(C_v)$  is

$$((c_1(F) - (\text{rk } F)(\gamma'_0 + s' H')) \cdot H') = 0.$$

We note that  $\text{rk } F = -\langle v, v_0 \rangle < 0$ . Then we have the following.

- (i)  $\sigma_{(\gamma'_0+s'H',t'H')} \in U'_+$  if and only if  $s' < 0$ .
- (ii)  $\sigma_{(\gamma'+s'H',t'H')} \in U'_-$  if and only if  $s' > 0$ .

Applying Proposition 3.11, we get a generalization of [10, Thm. 1.2].

**Theorem 3.12.** *Let  $X$  be a K3 surface or an Enriques surface over  $k$ . Let  $v$  be a Mukai vector with  $\langle v^2 \rangle \geq 0$ .*

- (1) *If  $(\gamma + sH, tH) \in U_+$ , then we have an isomorphism*

$$(3.11) \quad \begin{array}{ccc} \mathcal{M}_{(\gamma+sH,tH)}(v) & \rightarrow & \mathcal{M}_{H'}^{\gamma'}(v') \\ E & \mapsto & \Phi_{X \rightarrow X'}^{\mathcal{E}_+^\vee[1]}(E). \end{array}$$

- (2) *If  $(\gamma + sH, tH) \in U_-$ , then we have an isomorphism*

$$(3.12) \quad \begin{array}{ccc} \mathcal{M}_{(\gamma+sH,tH)}(v) & \rightarrow & \mathcal{M}_{H'}^{-\gamma'}(v'^\vee) \\ E & \mapsto & (\Phi_{X \rightarrow X'}^{\mathcal{E}_-^\vee[1]}(E))^\vee. \end{array}$$

The following result is a slight generalization of [11, Thm. 7.6].

**Theorem 3.13.** *Let  $X$  be a classical Enriques surface and  $v$  be a Mukai vector with  $\langle v^2 \rangle \geq 0$ . For a general  $\sigma$  with respect to  $v$ , there is a nef and big divisor  $H$  and  $\gamma \in \text{NS}(X)_\mathbb{Q}$  such that  $\mathcal{M}_\sigma(v)$  is isomorphic to  $\mathcal{M}_H^\beta(w)$ . In particular, there is a projective moduli space  $M_\sigma(v)$  of  $S$ -equivalence classes of  $\sigma$ -semi-stable objects  $E$  with  $v(E) = v$ .*

*Proof.* We may assume that  $\sigma = \sigma_{(\beta_0, \omega_0)}$  ( $(\beta_0, \omega_0) \in \text{NS}(X)_\mathbb{R} \times \text{Amp}(X)_\mathbb{R}$ ) and  $\text{Im } Z_{(\beta_0, \omega_0)}(v) > 0$ . We take  $E \in \mathcal{M}_{(\beta_0, \omega_0)}(v)$ . By perturbing  $(\beta_0, \omega_0)$ , we can find a primitive and isotropic Mukai vector  $v_0 := lre^{\xi/r}$  such that  $r$  is odd and

$$(3.13) \quad (\beta_0, \omega_0) \in W := \{(\beta, \omega) \mid \phi_{(\beta, \omega)}(E) = \phi_{(\beta, \omega)}(v_0)\}.$$

We set  $r_0 := lr$  and  $\gamma := \xi/r$ . Applying Theorem 3.12, we get our claim.  $\square$

**Remark 3.14.** (1) In the proof of Theorem 3.13, by perturbing  $(\beta_0, \omega_0)$  with the condition (3.13), we may assume that  $(\beta_0, \omega_0)$  is general with respect to  $v_0$  or  $W$  is the unique wall for  $v_0$  in a neighborhood of  $(\beta_0, \omega_0)$ . Then  $\omega'_0$  is ample. Hence we can take  $H$  to be ample.

- (2) If all divisor classes on  $X$  are defined, then by Remark 3.4, the same claim holds even if  $k$  is not algebraically closed.

**Remark 3.15.** Assume that  $X$  is a K3 surface. Then the same proof also works. In this case, the result was obtained in [10] under the assumption  $\rho(X) = 1$  and in [2] for the general case combining a classification of walls for  $\varrho_X$  with the argument of [10]. We also remark that the same proof of [2] also work for the case of an Enriques surface.

**3.5. Examples of isomorphisms.** Assume that  $X$  is a K3 surface with  $\text{Pic}(X) = \mathbb{Z}H$ . Let  $I_Z$  be an ideal sheaf with  $v(I_Z) = (1, 0, -n)$ . Then  $\phi_{(sH, tH)}(e^{\lambda H}) = \phi_{(sH, tH)}(I_Z)$  if and only if

$$(3.14) \quad t^2 + (s - \lambda) \left( s - \frac{2n}{(H^2)\lambda} \right) = 0.$$

We note that  $(-1, \sqrt{\frac{2}{(H^2)}})$  satisfies

$$t^2 + (s - \lambda) \left( s - \frac{2n}{(H^2)\lambda} \right) \leq 0$$

if and only if

$$(3.15) \quad (\lambda + 1) \left( \lambda \frac{(H^2)}{2} + n \right) + \lambda \geq 0.$$

Under this condition, we shall consider a Fourier-Mukai transform  $\Phi_{X \rightarrow Y}^{\mathcal{E}^\vee[1]}$ , where  $Y = M_{(sH, t'H)}(r_0 e^{\lambda H})$ ,  $t'$  is sufficiently close to  $t$  and  $\mathcal{E}$  is a universal family. By our assumption,  $t > \sqrt{\frac{2}{(H^2)}}$  on  $s = -1$ , and hence  $\mathcal{M}_{(sH, tH)}(v) = \mathcal{M}_H(v)$ .

For example, if  $-\lambda$  is a positive integer with  $-\lambda \geq \frac{2(n+2)}{(H^2)}$  and  $-\lambda \geq 2$ , then since  $f_{v_1}(s) \leq \sqrt{\frac{2}{(H^2)}}$  ([21, Rem. 1.15 (1)]),

$$\mathcal{M}_{(sH, tH)}(-e^{\lambda H}) = \{I_x^\vee(\lambda H)[1] \mid x \in X\}$$

for the point  $(-1, t)$  on the semi-circle (3.14). Thus  $Y = X$  and  $I_\Delta^\vee(\lambda H)[1] = \mathcal{E}$ . Therefore  $\Phi_{X \rightarrow Y}^{\mathcal{E}^\vee[1]} = \Phi_{X \rightarrow X}^{I_\Delta(-\lambda H)}$  and we get the following result.

**Proposition 3.16.** *Assume that  $-\lambda$  is a positive integer with  $-\lambda \geq \frac{2(n+2)}{(H^2)}$  and  $-\lambda \geq 2$ . Then  $(\Phi_{X \rightarrow X}^{I_\Delta(-\lambda H)}(I_Z))^\vee$  is a stable sheaf.*

*Proof.* We note that  $\phi_{(sH, tH)}(I_x^\vee(\lambda H)[1]) > \phi_{(sH, tH)}(I_Z)$  on the outside of the semi-circle. By Theorem 3.12 (2), we get the claim.  $\square$

Assume that  $(H^2) = n + 2$  and  $\lambda = -2$ , i.e.,  $-\lambda = \frac{2(n+2)}{(H^2)}$ . Then the semi-circle (3.14) passes at  $(-1, \sqrt{\frac{2}{(H^2)}})$  and defines a wall for  $v$ . Indeed there is an ideal sheaf  $I_Z$  fitting in the exact sequence

$$(3.16) \quad 0 \rightarrow \mathcal{O}_X(-H)^{\oplus 2} \rightarrow I_Z \rightarrow I_x^\vee(-2H)[1] \rightarrow 0$$

(see [13], [14]). If  $s < -1$ , then (3.14) is a wall. If  $n \geq 2$ , then there is an ideal sheaf  $I_Z$  fitting in the exact sequence

$$(3.17) \quad 0 \rightarrow F \rightarrow I_Z \rightarrow \mathcal{O}_X(-H)[1] \rightarrow 0$$

where  $F \in \mathcal{M}_H(2, -H, 2 - \frac{n}{2})$ , which gives a wall for  $s > -1$ . Moreover if  $n \geq 4$ , then all  $I_Z$  fits in an exact sequence

$$0 \rightarrow F' \rightarrow I_Z \rightarrow \mathcal{O}_X(-H)^{\oplus(\frac{n}{2}-2)}[1] \rightarrow 0$$

where  $F' \in \mathcal{M}_H(\frac{n}{2} - 1, -(\frac{n}{2} - 2)H, \frac{n^2}{4} - n - 4)$ .

By the Fourier-Mukai transform  $I_{X \rightarrow X}^{I_\Delta(2H)}$ , we have an exact sequence from (3.16):

$$0 \rightarrow E_0^{\oplus 2} \rightarrow E \rightarrow \mathcal{O}_x^\vee[1] \rightarrow 0,$$

where  $E_0 = I_{X \rightarrow X}^{I_\Delta(2H)}(\mathcal{O}_X(-H)) \in \mathcal{M}_H(\frac{n}{2} + 2, -H, 1)$ .  $E^\vee$  is a non-locally free sheaf with  $v(E^\vee) = (n + 4, 2H, 1)$ . We would like to remark that  $I_x^\vee(-2H)[1]$  is properly  $\sigma_{(sH, tH)}$ -semi-stable on (3.14) with  $s > -1$ . Indeed we set  $E_x := \Phi_{X \rightarrow X}^{I_\Delta^\vee[2]}(I_x^\vee(-H))$ . Then  $E_x$  is a stable locally free sheaf with  $v(E_x) = (\frac{n}{2} + 1, H, 1)$  and we have an exact triangle

$$(3.18) \quad E_x(-H) \rightarrow I_x^\vee(-2H)[1] \rightarrow \mathcal{O}_X(-H)[1]^{\oplus(\frac{n}{2}+2)} \xrightarrow{\varphi} E_x(-H)[1].$$

We shall prove that  $E_x(-H)$  and  $\mathcal{O}_X(-H)[1]$  are stable objects on (3.14) with  $s > -1$ . We note that  $\varphi$  is the evaluation map  $\mathcal{O}_X(-H)[1] \otimes H^0(E_x) \rightarrow E_x(-H)[1]$ . We set  $\Psi := \Phi_{X \rightarrow X}^{I_\Delta(H)} \circ \Phi_{X \rightarrow X}^{I_\Delta}$ . Then  $\Psi(E_x) = \mathcal{O}_x[-2]$ . Since  $\Psi(\mathcal{O}_X) = E_0[-2]$ ,

$$\Psi(I_x^\vee(-H)[1])[1] = \ker(E_0 \otimes \text{Hom}(E_0, \mathcal{O}_x) \rightarrow \mathcal{O}_x).$$

Since  $E_0$  and  $\mathcal{O}_x$  are  $\sigma_{(sH, tH)}$ -stable on  $s = -\frac{2}{n+4}$  and  $t > \frac{2}{n+4} \sqrt{\frac{2}{n+2}}$ ,  $E_x(-H)$  and  $\mathcal{O}_X(-H)[1]$  are  $\sigma_{(sH, tH)}$ -stable objects on (3.14) with  $s > -1$ .

By the Fourier-Mukai transform  $I_{X \rightarrow X}^{I_\Delta(2H)}$ , (3.18) is transformed to the exact triangle

$$(3.19) \quad F_x \rightarrow \mathcal{O}_x[-1] \rightarrow E_0^{\oplus(\frac{n}{2}+2)}[1] \rightarrow F_x[1],$$

where  $F_x = T_{E_0}^{-1}(\mathcal{O}_x)[-1]$ . We also have an expression

$$F_x^\vee = \ker(E_0 \otimes \text{Hom}(E_0, \mathcal{O}_x) \rightarrow \mathcal{O}_x).$$

Thus (3.19) shows that the corresponding stability condition is the boundary of  $U(X)$  of type  $((E_0)_-)$ .

*Remark 3.17.* Since  $\phi_{(sH, tH)}(E_x) = \phi_{(sH, tH)}(I_Z)$  on (3.14) with  $-1 < s < -\frac{n}{n+2}$ , we can also apply Theorem 3.12 (1). Indeed  $\phi_{(sH, tH)}(E_x) < \phi_{(sH, tH)}(I_Z)$  for  $-1 < s < -\frac{n}{n+2}$  on the outside of the semi-circle. Let  $\mathbf{E}$  be the family  $\{E_x(-H) \mid x \in X\}$ . Then  $\Phi_{X \rightarrow X}^{\mathbf{E}^\vee[1]}(I_Z)$  is a stable sheaf. Since  $\Phi_{X \rightarrow X}^{\mathbf{E}^\vee[1]} = T_{E_0} \circ \Phi_{X \rightarrow X}^{I_\Delta(2H)}$ , the relation with the stable sheaf  $\Phi_{X \rightarrow X}^{I_\Delta(2H)}(I_Z)^\vee$  is given by  $\Phi_{X \rightarrow X}^{\mathbf{E}^\vee[1]}(I_Z) = T_{E_0}(\Phi_{X \rightarrow X}^{I_\Delta(2H)}(I_Z))$  (cf. [15, Thm. 2.3]).

#### 4. APPENDIX

**4.1. Modifications of some results in [4].** In this section, we shall explain similar technical results to those in [4] which are necessary to describe  $\text{Stab}^\dagger(X)$  for Enriques surfaces.

**Lemma 4.1.** *Then  $\otimes K_X$  acts trivially on  $\text{Stab}^\dagger(X)$ . In particular  $E$  is  $\sigma$ -stable if and only if  $E(K_X)$  is  $\sigma$ -stable.*

*Proof.* Let  $\sigma(K_X)$  be the stability condition induced by the action  $\otimes K_X : \mathbf{D}(X) \rightarrow \mathbf{D}(X)$ . Then

$$\text{Stab}(X, K_X) := \{\sigma \in \text{Stab}(X) \mid \sigma(K_X) = \sigma\}$$

is a closed subset of  $\text{Stab}(X)$ . By [3, Lem. 6.4], it is also an open subset. Indeed for  $\sigma \in \text{Stab}(X, K_X)$ , if  $f(\sigma, \tau) < 1/2$ , then  $f(\sigma(K_X), \tau(K_X)) = f(\sigma, \tau) < 1/2$  (see also [4, Lem. 2.3] for the definition of  $f(\sigma, \tau)$ ). Hence  $f(\tau(K_X), \tau) < 1$ . Since the central charge of  $\tau$  and  $\tau(K_X)$  are the same, we get the claim.

Moreover  $U(X)$  is contained. Hence  $\text{Stab}^\dagger(X)$  is also contained.  $\square$

**Lemma 4.2.** *If  $E \in \mathbf{D}(X)$  satisfies  $E(K_X) \cong E$ , then  $\text{rk } E$  is even.*

*Proof.* Since  $\det(E) \cong \det(E)(\text{rk } E K_X)$ ,  $(\text{rk } E)K_X = 0$ . Hence  $\text{rk } E$  is even.  $\square$

**Lemma 4.3.** *Assume that  $E \in \mathbf{D}(X)$  is  $\sigma$ -stable with  $\langle v(E)^2 \rangle < 0$ . Then we have the following.*

(i)

$$(4.1) \quad \langle v(E)^2 \rangle = \begin{cases} -1, & \text{rk } E \equiv 1 \pmod{2}, \\ -2, & \text{rk } E \equiv 0 \pmod{2}. \end{cases}$$

(ii) *If  $\langle v(E)^2 \rangle = -1$ , then  $\text{Ext}^1(E, E) = \text{Ext}^2(E, E) = 0$ .*

(iii) *If  $\langle v(E)^2 \rangle = -2$ , then  $\text{Ext}^1(E, E) = 0$  and  $\text{Ext}^2(E, E) \cong k$ .*

*Proof.* We set  $v(E) = (r, \xi, \frac{s}{2})$ . Then  $r, s \in \mathbb{Z}$ ,  $\xi \in \text{NS}(X)$  and  $r \equiv s \pmod{2}$ . Since  $\langle v(E)^2 \rangle = (\xi^2) - rs$ ,  $\langle v(E)^2 \rangle$  is even if and only if  $r$  is even.

Assume that  $r$  is odd. Then  $\text{Hom}(E, E(K_X)) = 0$ . Indeed if there is a non-zero map  $E \rightarrow E(K_X)$ , then it is isomorphic by the stability of  $E$  and  $E(K_X)$ , which contradicts Lemma 4.2. Then

$$0 > \langle v(E)^2 \rangle = \dim \text{Ext}^1(E, E) - 1 \geq -1$$

implies  $\text{Ext}^1(E, E) = 0$  and  $\langle v(E)^2 \rangle = -1$ .

Assume that  $r$  is even. Then  $0 > \langle v(E)^2 \rangle$  means  $\langle v(E)^2 \rangle \leq -2$ . Then we see that

$$(4.2) \quad \begin{aligned} -2 &\geq \langle v(E)^2 \rangle = \dim \text{Ext}^1(E, E) - 1 - \dim \text{Hom}(E, E(K_X)) \\ &\geq -1 - \dim \text{Hom}(E, E(K_X)) \geq -2. \end{aligned}$$

Hence  $\langle v(E)^2 \rangle = -2$ ,  $\text{Ext}^1(E, E) = 0$  and  $\text{Hom}(E, E(K_X)) \cong k$ .  $\square$

**Definition 4.4.** (1) For a spherical object  $A$ ,  $T_A$  denotes the twist functor.

(2) An object  $B$  is exceptional, if  $\text{Hom}(B, B) = k$  and  $\text{Ext}^i(B, B) = 0$  for  $i \neq 0$ . For an exceptional object  $B$ , we have an autoequivalence  $T_B := \Phi_{X \rightarrow X}^{\mathcal{E}}$  of  $\mathbf{D}(X)$ , where

$$\mathcal{E} := \text{Cone}(B \boxtimes B^\vee \oplus B(K_X) \boxtimes (B(K_X))^\vee \rightarrow \mathcal{O}_\Delta).$$

**Proposition 4.5** ([4, Thm. 12.1]). *Let  $\sigma = (Z, \mathcal{P}) \in \partial U(X)$  be a general point of the boundary. Then exactly one of the conditions  $(A^+), (A^-), (C_k)$  in [4, Thm. 12.1] or the following conditions holds.*

$(B^+)$  *There is a rank  $r$  simple and rigid vector bundle  $B$  with  $\langle v(B)^2 \rangle = -1$  such that  $B$ ,  $B(K_X)$  and  $T_B(\mathcal{O}_x)$  are the stable factors of  $\mathcal{O}_x$  and the Jordan-Hölder filtration is*

$$0 \rightarrow B^{\oplus r} \oplus B(K_X)^{\oplus r} \rightarrow \mathcal{O}_x \rightarrow T_B(\mathcal{O}_x) \rightarrow 0.$$



( $B^-$ ) There is a rank  $r$  simple and rigid vector bundle  $B$  with  $\langle v(B)^2 \rangle = -1$  such that  $B$ ,  $B(K_X)$  and  $T_B^{-1}(\mathcal{O}_x)$  are the stable factors of  $\mathcal{O}_x$  and the Jordan-Hölder filtration is

$$0 \rightarrow T_B^{-1}(\mathcal{O}_x) \rightarrow \mathcal{O}_x \rightarrow (B^{\oplus r} \oplus B(K_X)^{\oplus r})[2] \rightarrow 0.$$

For the proof, we need a modification of [4, Lem. 12.2].

**Lemma 4.6.** *Let  $\sigma = (Z, \mathcal{P})$  be a stability condition on  $X$  and  $E \in \mathcal{P}(1)$  a semi-stable object of phase 1 such that  $E(K_X) \cong E$ .*

- (1) *If  $\text{Ext}^1(E, E) = 0$ , then any stable factor  $F$  of  $E$  satisfies  $\text{Ext}^1(F, F) = 0$ .*
- (2) *If  $\text{Ext}^1(E, E) \cong k^{\oplus 2}$ , then there is a stable factor  $A \in \mathcal{P}(1)$  satisfying*
  - (a)  *$\text{Ext}^1(E, E) = 0$  and*
  - (b)  *$\text{Hom}(A, E) \neq 0$  or  $\text{Hom}(E, A) \neq 0$ .*

*Proof.* Let  $F$  be a stable factor of  $E$ . Then there is an exact sequence

$$0 \rightarrow F' \rightarrow E \rightarrow G \rightarrow 0$$

in  $\mathcal{P}(1)$  such that  $F'$  is a successive extension of  $F$  and  $F(K_X)$ , and  $\text{Hom}(F, G) = \text{Hom}(F(K_X), G) = 0$ . Since  $\text{Hom}(F', G) = \text{Hom}(F', G(K_X)) = 0$  and  $E(K_X) \cong E$ , we see that  $F'(K_X) \cong F'$  and  $G(K_X) \cong G$ . Then we get

$$(4.3) \quad \dim \text{Ext}^1(F', F') + \dim \text{Ext}^1(G, G) \leq \dim \text{Ext}^1(E, E) \leq 2.$$

(1) If  $\text{Ext}^1(E, E) = 0$ , then  $\text{Ext}^1(F', F') = 0$ , which implies  $\langle v(F'), v(F') \rangle < 0$ . Since  $v(F') \in \mathbb{Z}v(F)$ , we also have  $\langle v(F)^2 \rangle < 0$ . Then  $\text{Ext}^1(F, F) = 0$  by Lemma 4.3. If  $\text{rk } F$  is odd, then  $\text{Ext}^2(F, F) = 0$  and if  $\text{rk } F$  is even, then  $\text{Ext}^2(E, E) \cong k$ . A similar claim also holds for  $F(K_X)$ .

Since  $G$  also satisfies the assumption of (1), inductively we get the claim for stable factors of  $G$ .

(2) We note that  $\text{rk } F'$  and  $\text{rk } G$  are even by Lemma 4.2. Hence  $\langle v(F')^2 \rangle$  and  $\langle v(G)^2 \rangle$  are even. Since

$$(4.4) \quad \dim \text{Ext}^2(F', F') = \dim \text{Hom}(F', F'), \quad \dim \text{Ext}^2(G, G) = \dim \text{Hom}(G, G),$$

$\dim \text{Ext}^1(F', F')$  and  $\dim \text{Ext}^1(G, G)$  are even. Therefore  $\text{Ext}^1(F', F') = 0$  or  $\text{Ext}^1(G, G) = 0$ . Applying (1), we get the claim.  $\square$

**4.2. A complement on the wall crossing in [9].** Let  $P_{\gamma, H}$  be a family of stability condition in (2.9). We shall study the wall crossing in [9] by using the description of stability conditions in section 2. For simplicity, we assume that  $X$  is a K3 surface. Similar claims also hold for the case of an Enriques surface. Let  $U$  be the open subset of  $P_{\gamma, H}$  such that  $(\beta, \omega) \in U$  if and only if  $Z_{(\beta, \omega)}(u) \notin \mathbb{R}_{\leq 0}$  for any  $u \in \Delta(X)$  with  $\text{rk } u > 0$ .

For a Mukai vector  $v = r + \xi + a\varrho_X$  ( $r \in \mathbb{Z}_{>0}, \xi \in \text{NS}(X), a \in \mathbb{Q}$ ), we set  $\delta := \frac{\xi}{r}$ . Then  $v = re^\delta - \frac{\langle v^2 \rangle}{2r} \varrho_X$ .

As in [18], we set

$$(4.5) \quad \begin{aligned} \xi(\beta, \omega) &:= \xi(\beta, \omega, 1)/r \\ &= e^\gamma \left( \frac{(\omega^2) - ((\beta - \delta)^2)}{2} + \frac{\langle v^2 \rangle}{2r^2} \right) \omega \\ &\quad + e^\gamma ((\beta - \delta) \cdot \omega)(\beta - \delta) + ((\beta - \delta) \cdot \omega) \left( e^\delta + \frac{\langle v^2 \rangle}{2r^2} \varrho_X \right) \in C^+(v) \end{aligned}$$

for  $(\beta, \omega) \in \text{NS}(X)_{\mathbb{R}} \times P^+(X)_{\mathbb{R}}$ , where  $P^+(v)$  is the positive cone of  $v^\perp$ , and  $C^+(v) := P^+(v)/\mathbb{R}_{>0}$ . We have

$$\xi(\beta, \omega) = \text{Im} \frac{e^{\beta + \sqrt{-1}\omega}}{Z_{(\beta, \omega)}(v)} \in C^+(v).$$

For  $v_1 \in H^*(X, \mathbb{Q})_{\text{alg}}$ ,  $Z_{(\beta, \omega)}(v_1) \in \mathbb{R}Z_{(\beta, \omega)}(v)$  if and only if  $\xi(\beta, \omega) \in v_1^\perp$ . For the open set  $U$ , semi-stability is constant on the fiber of  $\xi$  [20, Cor. 3.6]. We shall slightly generalize the result to a point of the boundary of  $U$ . Let  $(\beta, t_0 H)$  be a point of  $\partial U$  such that  $\beta \in \text{NS}(X)_{\mathbb{Q}}$  and  $Z_{(\beta, t_0 H)}(u) \neq 0$  for all  $u \in \Delta(X)$  with  $\text{rk } u > 0$ .

By Proposition 2.2,  $\sigma_{(\beta, t_0 H)}$ -semi-stability is equivalent to  $\sigma_{(\beta + sH, tH)}$ -semi-stability for  $(\beta + sH, tH) \in \xi^{-1}(\xi(\beta, t_0 H)) \cap U$  and  $s < 0$  ([20, Cor. 3.6]).

We next consider a point  $(\beta, t_0 H) \in \partial U$  such that  $Z_{(\beta, t_0 H)}(E) = 0$  for a  $\beta$ -twisted stable object  $E$ . Let  $\mathfrak{S}$  be the set of  $\beta$ -twisted semi-stable objects  $E$  with  $Z_{(\beta, t_0 H)}(E) = 0$ , and let  $\mathfrak{E} = \{G_1, \dots, G_n\}$  be the set of  $\beta$ -twisted stable objects  $G_i \in \mathfrak{S}$ .

**Lemma 4.7.** *There is a positive number  $\epsilon$  such that  $G_i$  are  $\sigma_{(\beta + sH, tH)}$ -stable for all  $0 > s \geq -\epsilon$  and  $t_- \leq t_0 \leq t_+$ . Moreover  $M_{(\beta + sH, tH)}(v(G_i))^{ss} = \{G_i\}$ .*

*Proof.*  $G_i \in \mathcal{A}_{(\beta, t_- H)}$  are  $\sigma_{(\beta, t_- H)}$ -stable and  $G_i \in \mathcal{A}_{(\beta, t_+ H)}[-1]$  are  $\sigma_{(\beta, t_+ H)}$ -stable. Hence there is a positive number  $\epsilon$  such that  $G_i$  are  $\sigma_{(\beta+sH, t_\pm H)}$ -stable for all  $0 > s \geq -\epsilon$ . If  $t_- \leq t_0 \leq t_+$ , then  $G_i$  are  $\sigma_{(\beta+sH, t_0 H)}$ -stable for all  $0 < s \leq \epsilon$ . Assume that  $G$  is a  $\sigma_{(\beta+sH, t_0 H)}$ -semi-stable object with  $v(G) = v(G_i)$ . Since  $\chi(G, G_i) > 0$ , there is a morphism  $\psi_1 : G \rightarrow G_i$  or a morphism  $\psi_2 : G_i(K_X) \rightarrow G$ . Since the phase are the same,  $\psi_1$  is injective and  $\psi_2$  is surjective. Since  $v(G) = v(G_i)$ ,  $\psi_1, \psi_2$  are isomorphisms. Therefore our claim holds.  $\square$

**Definition 4.8** ([9, Defn. 4.2.1]). We take  $t_+ > t_0 > t_-$  such that  $t_+ - t_-$  is sufficiently small.

- (1)  $E \in \mathcal{A}_{(\beta, t_- H)}$  is  $\sigma_{(\beta, t_0 H)}$ -semi-stable, if  $\phi_{(\beta, t_0 H)}(E_1) \leq \phi_{(\beta, t_0 H)}(E)$  for any proper subobject  $E_1 \neq 0$  of  $E$  with  $Z_{(\beta, t_0 H)}(E_1) \neq 0$ . If  $\phi_{(\beta, t_0 H)}(E_1) < \phi_{(\beta, t_0 H)}(E)$  for any proper subobject  $E_1 \neq 0$  of  $E$  with  $Z_{(\beta, t_0 H)}(E_1) \neq 0$ , then  $E$  is  $\sigma_{(\beta, t_0 H)}$ -stable.
- (2) Let  $\mathcal{M}_{(\beta, t_0 H)}(v)$  (resp.  $\mathcal{M}_{(\beta, t_0 H)}(v)^s$ ) be the moduli stack of  $\sigma_{(\beta, t_0 H)}$ -semi-stable objects (resp.  $\sigma_{(\beta, t_0 H)}$ -stable objects)  $E$  with  $v(E) = v$ .

*Remark 4.9.* If there is a homomorphism  $\psi : E \rightarrow G_i$  for a  $\sigma_{(\beta, t_0 H)}$ -semi-stable object  $E$ , then  $\psi$  is surjective and  $\phi_{(\beta, t_0 H)}(\ker \psi) = \phi_{(\beta, t_0 H)}(E)$ . Hence  $\text{Hom}(E, G_i) = 0$  for a  $\sigma_{(\beta, t_0 H)}$ -stable object  $E$ .

In order to relate  $\sigma_{(\beta, t_0 H)}$ -semi-stability with Bridgeland semi-stability, we first prove the following.

**Lemma 4.10.** Assume that  $Z_{(\beta, t_0 H)}(v) \in \mathbb{R}_{>0} e^{\pi\sqrt{-1}\phi}$ ,  $0 < \phi < 1$ . Then

$$(4.6) \quad \mathcal{M}_{(\beta, t_+ H)}(v)^s \cap \mathcal{M}_{(\beta, t_- H)}(v)^s = \mathcal{M}_{(\beta, t_0 H)}(v)^s.$$

*Proof.* Assume that  $E \in \mathcal{M}_{(\beta, t_+ H)}(v)^s \cap \mathcal{M}_{(\beta, t_- H)}(v)^s$ . If  $E$  is not  $\sigma_{(\beta, t_0 H)}$ -stable, then there is a subobject  $F$  in  $\mathcal{A}_{(\beta, t_- H)}$  such that  $\phi_{(\beta, t_0 H)}(F) = \phi_{(\beta, t_0 H)}(E)$ . We take an exact sequence in  $\mathcal{A}_{(\beta, t_- H)}$

$$(4.7) \quad 0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0$$

such that  ${}^p H^{-1}(F_1) = {}^p H^{-1}(F) \in \mathcal{F}_{(\beta, t_- H)}(\subset \mathcal{F}_{(\beta, t_+ H)})$ ,  ${}^p H^0(F_1) \in \mathcal{T}_{(\beta, t_+ H)}$  and  $F_2 \in \mathfrak{S}(\subset \mathcal{T}_{(\beta, t_- H)})$ . Then  $E/F_1 \in \mathcal{A}_{(\beta, t_- H)}$ . Since  $\text{Hom}(E, G) = 0$  for  $G \in \mathfrak{S}$ , we have  $\text{Hom}(E/F_1, G) = 0$  for  $G \in \mathfrak{S}$ , which implies  $E/F_1 \in \mathcal{A}_{(\beta, t_+ H)}$  by [9, Lem. 4.2.2]. Since  $F_1 \in \mathcal{A}_{(\beta, t_+ H)}$ , we have an exact sequence in  $\mathcal{A}_{(\beta, t_\pm H)}$ :

$$(4.8) \quad 0 \rightarrow F_1 \rightarrow E \rightarrow E/F_1 \rightarrow 0.$$

By the stability of  $E$ , we have  $\phi_{(\beta, t_\pm H)}(F_1) \leq \phi_{(\beta, t_\pm H)}(E)$ . Since  $\phi_{(\beta, t_0 H)}(F_1) = \phi_{(\beta, t_0 H)}(F) = \phi_{(\beta, t_0 H)}(E)$ , we have  $\phi_{(\beta, t_\pm H)}(F_1) = \phi_{(\beta, t_\pm H)}(E)$ , which means  $E$  is properly  $\sigma_{(\beta, t_\pm H)}$ -semi-stable. Therefore  $E \in \mathcal{M}_{(\beta, t_0 H)}(v)^s$ .

Conversely for  $E \in \mathcal{M}_{(\beta, t_0 H)}(v)^s$ , assume that  $E \notin \mathcal{M}_{(\beta, t_+ H)}(v)^s \cap \mathcal{M}_{(\beta, t_- H)}(v)^s$ . If  $E \notin \mathcal{M}_{(\beta, t_+ H)}(v)^s$ , then there is a subobject  $F$  of  $E$  in  $\mathcal{A}_{(\beta, t_+ H)}$  such that  $\phi_{(\beta, t_+ H)}(F) \geq \phi_{(\beta, t_+ H)}(E)$ . Then  $\phi_{(\beta, t_0 H)}(F) \geq \phi_{(\beta, t_0 H)}(E)$ . For  $E/F \in \mathcal{A}_{(\beta, t_+ H)}$ , we have an exact sequence in  $\mathcal{A}_{(\beta, t_+ H)}$

$$(4.9) \quad 0 \rightarrow F' \rightarrow E/F \rightarrow F_2 \rightarrow 0$$

such that  $F' \in \mathfrak{S}[1]$ ,  ${}^p H^{-1}(F_2) \in \mathcal{F}_{(\beta, t_- H)}$  and  ${}^p H^0(E/F) = {}^p H^0(F_2) \in \mathcal{T}_{(\beta, t_+ H)}(\subset \mathcal{T}_{(\beta, t_- H)})$ . Then  $E \rightarrow F_2$  is surjective in  $\mathcal{A}_{(\beta, t_+ H)}$ . We set  $F_1 = \ker(E \rightarrow F_2) \in \mathcal{A}_{(\beta, t_+ H)}$ . By the construction of  $F_2$ , we have  $F_2 \in \mathcal{A}_{(\beta, t_- H)}$ . Since  $\text{Hom}(G[1], E) = 0$  for  $G \in \mathfrak{S}$ , we have  $\text{Hom}(G[1], F_1) = 0$ . Hence  $F_1 \in \mathcal{A}_{(\beta, t_- H)}$ . Thus we have an exact sequence

$$0 \rightarrow F_1 \rightarrow E \rightarrow F_2 \rightarrow 0$$

in  $\mathcal{A}_{(\beta, t_\pm H)}$ . Then  $\phi_{(\beta, t_0 H)}(F_1) = \phi_{(\beta, t_0 H)}(F) \geq \phi_{(\beta, t_0 H)}(E)$ , which shows that  $E$  is not  $\sigma_{(\beta, t_0 H)}$ -stable.

If  $E \notin \mathcal{M}_{(\beta, t_- H)}(v)^s$ , then there is a subobject  $F$  of  $E$  in  $\mathcal{A}_{(\beta, t_- H)}$  such that  $\phi_{(\beta, t_- H)}(F) \geq \phi_{(\beta, t_- H)}(E)$ . Then we have  $\phi_{(\beta, t_0 H)}(F) \geq \phi_{(\beta, t_0 H)}(E)$ , which shows that  $E$  is not  $\sigma_{(\beta, t_0 H)}$ -stable. Therefore we get our claim.  $\square$

**Proposition 4.11.**  $\mathcal{M}_{(\beta', \omega')}(v) = \mathcal{M}_{(\beta, t_0 H)}(v)$  for  $(\beta', \omega') \in \xi^{-1}(\xi(\beta, t_0 H))$  with  $(\beta', \omega') = (\beta + sH, tH)$ ,  $-\epsilon < s < 0$ .

*Proof.* For  $E \in \mathcal{M}_{(\beta, t_0 H)}(v)$ , we have a filtration

$$(4.10) \quad 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$$

such that  $E_i := F_i/F_{i-1}$  are  $\sigma_{(\beta, t_0 H)}$ -stable and  $Z_{(\beta, t_0 H)}(E_i) = \lambda_i Z_{(\beta, t_0 H)}(E)$  with  $0 \leq \lambda_i \leq 1$ . If  $\lambda_i = 0$ , then  $E_i \in \mathfrak{E}$ . We set  $v_i := v(E_i)$ . Then  $\langle v_i^2 \rangle \geq -2$  and  $\xi(\beta, t_0 H) \in v_i^\perp$ . (4.10) is the Jordan-Hölder filtration of  $E$  with respect to  $\sigma_{(\beta, t_0 H)}$ . For the proof of our claim, it is sufficient to prove that

$$(4.11) \quad \mathcal{M}_{(\beta', \omega')}(v_i)^s = \mathcal{M}_{(\beta, t_0 H)}(v_i)^s$$

for  $(\beta', \omega') \in \xi^{-1}(\xi(\beta, t_0 H))$  with  $\beta' = \beta + sH$ ,  $s < 0$  and any decomposition  $v = \sum_i v_i$  of  $v$  such that  $\xi(\beta, t_0 H) \in v_i^\perp$  with  $\langle v_i^2 \rangle \geq -2$ . Indeed (4.11) means that semi-stability and its  $S$ -equivalence class with

respect to  $\sigma_{(\beta', \omega')}$  is the same as those for  $\sigma_{(\beta, t_0 H)}$ . We take  $t_{\pm}$  such that  $t_- < t_0 < t_+$  and  $t_+ - t_- \ll 1$ . By Lemma 4.10, we have

$$(4.12) \quad \mathcal{M}_{(\beta, t_+ H)}(v_i)^s \cap \mathcal{M}_{(\beta, t_- H)}(v_i)^s = \mathcal{M}_{(\beta, t_0 H)}(v_i)^s.$$

We first assume that  $\lambda_i \neq 0$ . As in [20, Rem. 3.6], we set  $\xi_{v_i}(\beta', \omega') := \text{Im}(Z_{(\beta', \omega')}(v_i)^{-1} e^{\beta' + i\omega'})$ . Since  $\text{Im}(Z_{(\beta, t_0 H)}(v_i)^{-1} e^{\beta + it_0 H}) = \lambda_i^{-1} \text{Im}(Z_{(\beta, t_0 H)}(v_i)^{-1} e^{\beta + it_0 H})$ , we get  $\xi_{v_i}(\beta, t_0 H) \in \mathbb{R}\xi(\beta, t_0 H)$ . By [20, Rem. 3.6],  $\xi_{v_i}^{-1}(\xi_{v_i}(\beta, t_0 H)) = \xi^{-1}(\xi(\beta, t_0 H))$ . We take  $(\beta'_{\pm}, \omega'_{\pm}) \in \xi_{v_i}^{-1}(\xi_{v_i}(\beta, t_{\pm} H))$  which are in a neighborhood of  $(\beta', \omega')$ . Then  $\mathcal{M}_{(\beta, t_{\pm} H)}(v_i)^s = \mathcal{M}_{(\beta'_{\pm}, \omega'_{\pm})}(v_i)^s$ . Since

$$\mathcal{M}_{(\beta'_{-}, \omega'_{-})}(v_i)^s \cap \mathcal{M}_{(\beta'_{+}, \omega'_{+})}(v_i)^s = \mathcal{M}_{(\beta', \omega')}(v_i)^s,$$

we get (4.11). If  $E_i = G_j$ , then it is also  $\sigma_{(\beta', \omega')}$ -stable with  $Z_{(\beta', \omega')}(E_i) \in \mathbb{R}_{>0} Z_{(\beta', \omega')}(v)$  by Lemma 4.7. Hence  $E$  is  $\sigma_{(\beta', \omega')}$ -semi-stable with a Jordan-Hölder filtration (4.10).  $\square$

**Corollary 4.12.** *If  $(\beta, t_0, H)$  belongs to any wall with respect to  $\sigma$ , that is, there is  $E_1 \oplus E_2 \in \mathcal{M}_{(\beta, t_0 H)}(v)$  such that  $v(E_1) \notin \mathbb{Q}v(E_2)$ , then  $(\beta, t_+ H)$  and  $(\beta, t_- H)$  belong to the same chamber, where  $t_+ > t_0 > t_-$  and  $t_+ - t_- \ll 1$ .*

Assume that  $v = r + \xi + b\rho_X \in v(K(X))$  satisfies

$$((\xi - r\beta) \cdot H) = \min\{((c_1(F) - \text{rk } F\beta) \cdot H) > 0 \mid F \in K(X)\}.$$

Then  $(\beta, t_0 H)$  lies on a wall for  $v$  if and only if there is a decomposition  $v = \sum_i v_i$  such that  $Z_{(\beta, t_0 H)}(v_i) \in \mathbb{R}_{\geq 0} Z_{(\beta, t_0 H)}(v)$  and  $\langle v_i^2 \rangle \geq -2$ . We set  $v := e^{\beta}(r + dH + D + a\rho_X)$  and  $v_i := e^{\beta}(r_i + d_i H + D_i + a_i \rho_X)$  ( $D_i \in H^{\perp}$ ). Then  $d_i \geq 0$  for all  $i$  and  $\sum_i d_i = d$ . Hence we may assume that  $d_1 = d$  and  $d_i = 0$  for  $i \geq 2$ . Then  $Z_{(\beta, t_0 H)}(v_i) = 0$  for  $i \geq 2$ . By  $\langle v_i^2 \rangle \geq -2$ ,  $v_i$  ( $i \geq 2$ ) satisfy

$$(4.13) \quad \begin{aligned} \langle v_i^2 \rangle &= -2, \quad (i \geq 2) \\ \langle v^2 \rangle - 2\langle v, \sum_{i \geq 2} v_i \rangle + \langle (\sum_{i \geq 2} v_i)^2 \rangle &\geq -2. \end{aligned}$$

*Example 4.13.* Let  $X$  be a K3 surface with  $\text{Pic}(X) = \mathbb{Z}H$ . Let  $U$  be an exceptional vector bundle with  $v(U) = e^{\beta}(r_0 + \frac{1}{r_0}\rho_X)$ . If  $v = -e^{\beta}(r - dH + a\rho_X)$ , then (4.13) is  $r/r_0 + ar_0 \leq \langle v^2 \rangle/2$ , where  $v_1 = v - v(U)$  and  $v_2 = v(U)$ . In particular if  $r/r_0 + ar_0 > \langle v^2 \rangle/2$ , then there is no wall in  $\{(\beta + sH, tH) \mid s \leq 0\}$ .

We shall see the wall by the computation in [9]. We set  $t_0 := \frac{1}{r_0} \sqrt{\frac{2}{(H^2)}}$ . Then  $t = t_0$  is the candidate of a unique wall on the half line  $\{(\beta + sH, tH) \mid s = 0\}$ . Let  $t_-, t_+$  be numbers with  $t_- < t_0 < t_+$ . We take  $E := F[1] \in \mathcal{M}_{(\beta, t_+ H)}(v)$ . We note that  $F^{\vee}$  is a stable sheaf by [9, Cor. 3.2.1]. If  $\text{Hom}(U, E) \neq 0$ , then we have a stable sheaf  $(F')^{\vee}$  fitting in the extension

$$0 \rightarrow U^{\vee} \rightarrow (F')^{\vee} \rightarrow F^{\vee} \rightarrow 0,$$

which gives an exact sequence

$$0 \rightarrow U \rightarrow F[1] \rightarrow F'[1] \rightarrow 0$$

in  $\mathcal{A}_{(\beta + sH, tH)}$  for  $s < 0$  (and  $t \gg 0$ ). The condition for the existence of  $F'$  is  $\langle v(F')^2 \rangle = \langle v^2 \rangle - 2(r/r_0 + ar_0) - 2 \geq -2$ . Therefore  $U$  defines a wall in  $s < 0$  if  $r/r_0 + ar_0 \leq \langle v^2 \rangle/2$ .

*Remark 4.14.* If  $\langle v, v(U) \rangle = r/r_0 + ar_0 \leq \langle v^2 \rangle/2$ , then  $U[1]$  defines a wall in  $\{(\beta + sH, tH) \mid s > 0\}$ . Indeed for  $E \in \mathcal{M}_{(\beta, t_+ H)}(v)$ , we have  $\text{Hom}(E, U[1]) \neq 0$ . Hence we have an exact sequence in  $\mathcal{A}_{(\beta, t_+ H)}$

$$(4.14) \quad 0 \rightarrow E' \rightarrow E \rightarrow U[1] \rightarrow 0,$$

where  $E'$  is also  $\sigma_{(\beta, t_+ H)}$ -stable. In the region  $\{(\beta + sH, tH) \mid s > 0\}$ , (4.14) gives a wall for  $v$ .

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